THE ONTARIO HIGH SCHOOL GEOMETRY

Old Edition

AUTHORIZED BY
THE MINISTER OF EDUCATION FOR ONTARIO

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The Student should in some way gain an insight into the methods employed by the author in solving his problem. - - - The only possible justification for giving a student a solution is to train him in methods of solving problems economically and in rigid form.

See: "The Teaching of Mathematics" by Schultze.

For use of N. P.

There is a large part of geometry which may be described as the explicit statements of characteristics which are capable of direct observation, but which are not distinguished from total experience until geometrical experience has made them clear. The attitude of the ordinary man to one of these explicit statements in geometry is hardly worth making because it is so obvious. —Judd.—

THE ONTARIO

HIGH SCHOOL GEOMETRY

THEORETICAL

BY

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Authorized by the Minister of Education for Ontario

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PREFACE

The Ontario High School Geometry is intended to cover the course in Theoretical Geometry, begun in the Lower School and completed in the Middle School, as defined in the Programme of Studies for High Schools and Collegiate Institutes of the Province of Ontario.

In deference to the wish of the teachers of mathematics of the Province, this Geometry is divided into Books with numbered propositions.

While the theoretical course is complete in itself, it is assumed that its study has been preceded by the usual course in drawing and measurement. A considerable number of practical problems are given in the exercises. These should be worked out carefully, and, in fact, all diagrams should be accurately and neatly made.

The book contains an abundant supply of carefully selected and graded exercises. Those given in sets throughout the Books will be found suitable for the work of average classes, and just about sufficient in number to fix the subject-matter of the propositions in the minds of the pupils. All the problems contained in the miscellaneous collections at the ends of the Books could be worked through by a few of the best pupils only, and should be used also by the teachers as a store from which to draw suitable material for review purposes from time to time.

While the requirements of class-work have been constantly kept in mind in the choice of proofs, it should not be assumed that other proofs, just as good, cannot in many cases be given.

Students should be constantly encouraged to work out methods of their own, and to keep records of the best in their note books.

Symmetry has been used to an unusual extent in giving a more concise form to the proofs of constructions.

The treatment of parallels, in accord with the method of many of the best English text-books, is based on Playfair's Axiom.

Tangents are treated both by the method of limits and as lines which meet the circle in only one point.

Areas of triangles and parallelograms are compared with rectangles, thereby not only giving a simple method of treatment, but also promoting facility in numerical computations.

Similarly, the treatment of proportion is correlated with the algebraic knowledge of the pupil.

OTTAWA, June, 1910.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used :—

Fig.	Figure.
Const.	Construction.
Hyp.	Hypothesis.
Cor.	Corollary.
e.g.	*exempli gratia*, for example.
i.e.	*id est*, that is.
p.	page.
$\because$	because, since.
$\therefore$	therefore.
rt.	right.
st.	straight.
$\angle$, $\angle$s, $\angle$d	angle, angles, angled.
$\triangle$, $\triangle$s	triangle, triangles.
$\parallel$, $\parallel$s	parallel, parallels.
$\parallel$gm, $\parallel$gms	parallelogram, parallelograms.
sq., sqs.	square, squares.
AB^2	the square on AB.
rect.	rectangle.
AB.CD	the rectangle contained by AB and CD.
AB : CD, or $\frac{AB}{CD}$	the ratio of AB to CD.
$+$	plus, together with.
$-$	minus, diminished by.
$\perp$	is perpendicular to, a perpendicular.
$=$	is equal to, equals.
$>$	is greater than.
$<$	is less than.
$\equiv$	is congruent to, congruent.
$\vert\vert\vert$	is similar to, similar.

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THEORETICAL GEOMETRY

BOOK I

Preliminary Definitions and Explanations

1. A **point** is that which has position but no size.

The position of a point on the blackboard, or on paper, is represented by a mark. This mark has some small size and therefore only roughly represents the idea of a point.

2. A **line** is that which has length but neither breadth nor thickness.

Again, the mark that we use to represent a line has breadth and some small thickness, and consequently, only roughly represents the idea.

The intersection of two lines is a point.

3. Lines may be either straight or curved.

The following property distinguishes straight lines from curved lines and may be used as the definition of a straight line:—

Two straight lines cannot have any two points of one coincide with two points of the other without the lines coinciding altogether.

This is sometimes stated as follows:—Joining two points there is always one and only one straight line.

It follows from this definition that **two straight lines cannot enclose a space.**

Can the circumferences of two equal circles coincide in two points without coinciding altogether?

4. A **surface** is that which has length and breadth but no thickness.

A sheet of tissue paper has length and breadth and very little thickness. It thus roughly represents the idea of a surface. In fact the sheet of paper has two well-defined surfaces separated by the substance of the paper.

The boundary between two parts of space is a surface.

5. Surfaces may be either plane or curved.

The following property distinguishes plane surfaces from curved surfaces and may be used as the definition of a plane surface:—

The straight line joining any two points on a plane surface lies wholly on that surface.

Give examples of curved surfaces on which straight lines may be drawn in certain directions. Notice the force of the word "any" in the definition above.

6. A **solid** is that which has length, breadth and thickness.

7. Any combination of points, lines, surfaces and solids is called a **figure.**

8. **Geometry** is the science which investigates the properties of figures and the relations of figures to one another.

9. In **Plane Geometry** the figure, or figures, considered in each proposition are confined to one plane, while **Solid Geometry** treats of figures the parts of which are not all in the same plane.

Plane Geometry is also called Geometry of Two Dimensions (length and breadth), and Solid Geometry is called Geometry of Three Dimensions (length, breadth and thickness).

Geometrical Reasoning

10. Two general methods of investigating the properties or relations of figures may be distinguished as the Practical Method and the Theoretical Method.

Some properties may be tested by measurement, paper-folding, etc., while in the same or other cases it may be shown that the property follows as a necessary result from others that are already known to be true.

The Theoretical Method, has certain advantages over the Practical method. Measurements, etc., are never exact, and in many cases cannot be made directly; but in the Theoretical Method, starting from certain simple statements, called **axioms,** the truth of which is self-evident, or, it may be in some cases, assumed, the consequent statements follow with absolute certainty.

The Practical Method is also known as the Inductive Method of Reasoning, and the Theoretical Method as the Deductive Method.

11. Figures may be compared by making a tracing of one of them and fitting the tracing on the other. In many cases the process may be made a mental operation and the comparison made with absolute certainty by means of the following axiom:—

A figure may be, actually or mentally, transferred from one position to another without change of form or size.

When two figures are shown to be exactly equal in all respects by supposing one to be made to fit exactly on the other, the proof is said to be by the **method of superposition.**

Figures which exactly fill the same space are said to **coincide** with each other.

12. In general a **proposition** is that which is stated or affirmed for discussion.

In mathematics a **proposition** is a statement of either a truth to be demonstrated or of an operation to be performed. It is called a **theorem** when it is something to be proved, and a **problem** when it is a construction to be made.

Example of Theorem:—If two straight lines cut each other, the vertically opposite angles are equal.

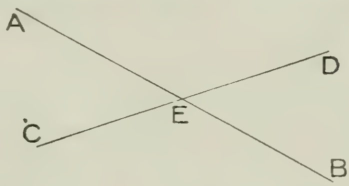
Example of Problem:—It is required to bisect a given straight line.

13. Theorems are commonly stated in two ways:—First, the **General Enunciation,** in which the property is stated as true for all figures of a class, but without naming any particular figure, as in the first example given in § 12; second, the **Particular Enunciation,** in which the theorem is stated to be true of the particular figure in a certain diagram.

Similarly general and particular enunciations are commonly given for problems.

Examples of Particular Enunciation :—

1. Let **AB** and **CD** be two st. lines cutting at **E**.



It is required to show that ∠ **AEC** = ∠ **BED**, and that ∠ **AED** = ∠ **BEC**.

2. Let **AB** be a given st. line.

It is required to bisect **AB**.

14. In general, the enunciation of a theorem consists of two parts: the hypothesis and the conclusion.

The **hypothesis** is the formal statement of the conditions that are supposed to exist, *e.g.*, in the first example of § 12, "If two straight lines cut each other."

The **conclusion** is that which is asserted to follow necessarily from the hypothesis, *e.g.*, "the vertically opposite angles are equal to each other."

Commonly, the hypothesis of a theorem is stated first, introduced by the word "if," and the two parts hypothesis and conclusion are separated by a comma. Sometimes, however, the two parts are not so formally

distinguished, *e.g.*, in the proposition:—The angles at the base of an isosceles triangle are equal to each other. In order to show the two parts, this statement may be changed as follows:—If a triangle has two sides equal to each other, the angles opposite these equal sides (or angles at the base) are equal to each other.

15. The demonstration of a theorem depends either on definitions and axioms, or on other theorems that have been previously shown to be true.

The following are some of the axioms commonly used in geometrical reasoning:—

1. Things that are equal to the same thing are equal to each other.

If A = B, B = C, C = D, D = E and E = F, what about A and F?

2. If equals be added to equals the sums are equal.

Thus if A, B, C, D be four st. lines such that A = B and C = D, then the sum of A and C = the sum of B and D.

Exercise:—Mark four successive points A, B, C, D on a st. line such that AB = CD. Show that AC = BD.

3. If equals be taken from equals the remainders are equal.

Give example.

Exercise:—Mark four successive points A, B, C, D on a st. line such that AC = BD. Show that AB = CD.

4. If equals be added to unequals the sums are unequal, the greater sum being obtained from the greater unequal.

Give example. Show also, by example, that if unequals be added to unequals the sums may be either equal or unequal.

5. If equals be taken from unequals the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Doubles of the same thing, or of equal things, are equal to each other.

7. Halves of the same thing, or of equal things, are equal to each other.

8. The whole is greater than its part, and equal to the sum of all its parts.

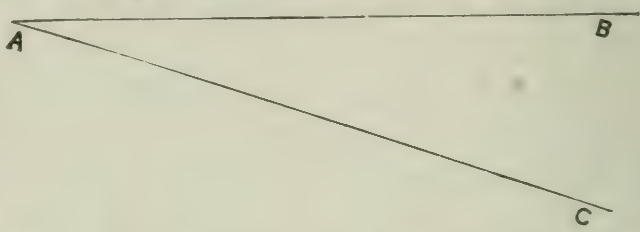
Give examples.

9. Magnitudes that coincide with each other, are equal to each other.

These simple propositions, and others that are also plainly true, may be freely used in proving theorems.

Angles and Triangles

16. **Definitions.** — When two straight lines are drawn from a point they are said to form an **angle.**



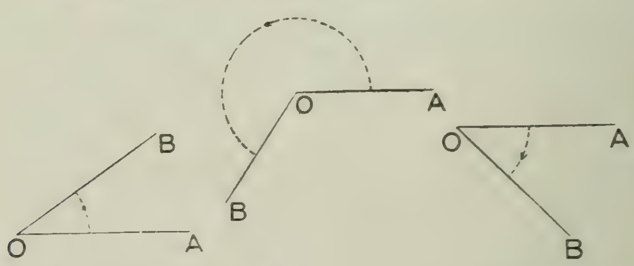
The point from which the two lines are drawn is called the **vertex** of the angle.

The two lines are called the **arms** of the angle.

The angle in the figure may be called the angle BAC, or the angle CAB. The letter at the vertex must be the middle one in reading the angle.

The single letter at the vertex is sometimes used to denote the angle when there can be no doubt as to which angle is meant.

17. Suppose a straight line OB to be fixed, like a rigid rod on a pivot at the point O, and be free to rotate in the plane of the paper.



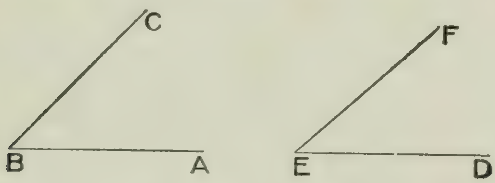
If the line OB start from any position OA, it may rotate in either of two directions—that in which the hands of a clock rotate, or in the opposite.

When OB starts from OA and stops at any position an angle is formed with O for its vertex and OA and OB for its arms.

18. An angle is said to be positive or negative according to the direction in which the line that traces out the angle is supposed to have rotated. The direction contrary to that in which the hands of a clock rotate is commonly taken as positive.

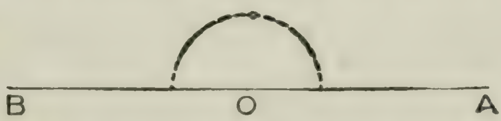
19. The magnitude of an angle depends altogether on the amount of rotation, and is quite independent of the lengths of its arms.

20. If we wish to compare two angles ABC and DEF we may suppose the angle ABC to be placed on



the angle DEF so that B falls on E and BA along ED. The position of BC with respect to EF will then show which of the angles is the greater and by how much it is greater than the other.

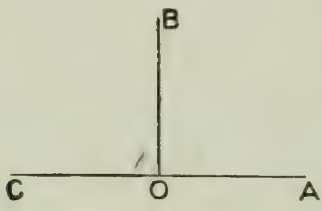
21. Definition.—When a revolving line OB has made half of a complete revolution from the initial position OA the angle formed is a **straight angle.**



The arms of a straight angle are thus in the same straight line and extend in opposite directions from

the vertex. At the point **O**, in the diagram, there are two straight angles on opposite sides of the straight line **AOB**, the two straight angles making up the complete revolution.

22. Definition.—If a straight line, starting from **OA**, rotates in succession through two equal angles **AOB**,



BOC, the sum of which is a straight angle, each of these angles is called a **right angle.**

A right angle is thus one-half of a straight angle, or one-quarter of a complete revolution.

Each arm of a right angle is said to be **perpendicular** to the other arm.

What is a *vertical* line? a *horizontal* line?

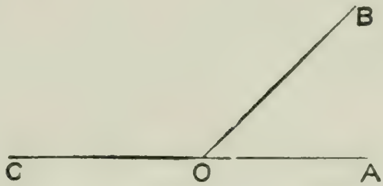
An angle which is less than a right angle is called an **acute angle.**

An angle which is greater than a right angle is called an **obtuse angle.**

23. If a right $\angle$ be divided into ninety equal parts, each of these parts is called a degree.

$$
\begin{aligned}
\text{Thus } 1 \text{ rt. } \angle &= 90^\circ, \\
1 \text{ st. } \angle &= 180^\circ \\
1 \text{ revolution} &= 360^\circ.
\end{aligned}
$$

24. Let a st. line starting from OA revolve through two successive ∠s



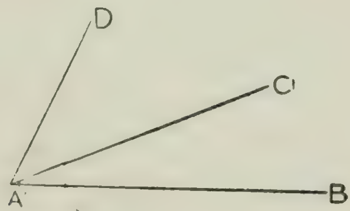
AOB, BOC such that OC is in the same st. line with OA, but in the opposite direction from the point O, and consequently AOC is a st. ∠.

$$\because \angle AOB + \angle BOC = \text{the st. } \angle AOC,$$

$$\therefore \angle AOB + \angle BOC = 2 \text{ rt. } \angle s.$$

Thus the angles which one straight line makes with another on the same side of that other are together equal to two right angles.

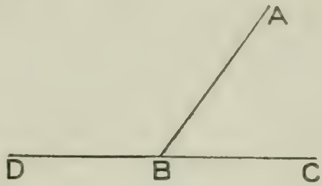
25. **Definition.**—When two angles have the same vertex and a common arm, and the remaining arms on opposite sides of the common arm, they are said to be **adjacent angles.**



Thus BAC and CAD are adjacent angles having the same vertex A and the common arm AC.

But angles BAD and CAD, with the same vertex and the common arm AD are not adjacent angles.

26. Let the adjacent $\angle$s ABC, ABD be together equal to two rt. $\angle$s.



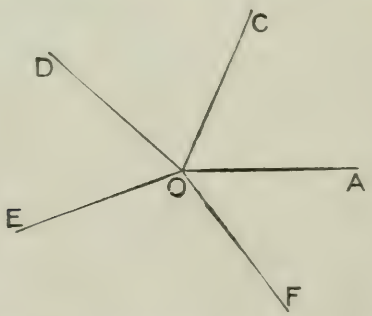
$\angle$ ABD + $\angle$ ABC = two rt. $\angle$s = a st. $\angle$.

That is, $\angle$ DBC is a st. $\angle$,

and $\therefore$ line DBC is a st. line.

Thus, if two adjacent angles are together equal to two right angles, the exterior arms of the angles are in the same straight line.

27. Let a st. line OB, starting from the position OA, and rotating in the positive direction, trace out the successive $\angle$s: AOC, COD, DOE, EOF, FOA.

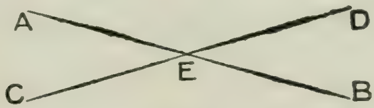


The sum of the successive $\angle$s is a complete revolution, and therefore equal to four rt. $\angle$s.

Thus, if any number of straight lines meet at a point, the sum of the successive angles is four right angles.

THEOREM 1

Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle.



Hypothesis.—The two st. lines AB, CD cut each other at E.

To prove that (1) $\angle$ AEC = $\angle$ BED,

(2) $\angle$ AED = $\angle$ BEC.

Proof.—$\because$ CED is a st. line,

$\angle$ AEC + $\angle$ AED = two rt. $\angle$s.

$\because$ AEB is a st. line,

$\angle$ AED + $\angle$ DEB = two rt. $\angle$s.

$\therefore$ $\angle$ AEC + $\angle$ AED = $\angle$ AED + $\angle$ DEB.

From each of these equals take away the common $\angle$ AED and the remainders must be equal to each other.

$\therefore$ $\angle$ AEC = $\angle$ DEB.

In the same manner it may be shown that $\angle$ AED = $\angle$ CEB.

---

28. **Definitions.**—When two angles are such that their sum is two right angles, they are said to be **supplementary** angles, or each angle is said to be the **supplement** of the other.

If two $\angle$s are equal, what about their supplementary $\angle$s?

When two angles are such that their sum is one right angle, they are said to be **complementary** angles, or each angle is said to be the **complement** of the other.

29.—Exercises

1. If one of the four $\angle$s made by two intersecting st. lines be 17°, find the number of degrees in each of the other three.

2. Two st. lines **ABD**, **CBE** cut at **B**, and $\angle$ **ABC** is a rt. $\angle$. Prove that the other $\angle$s at **B** are also rt. $\angle$s.

3. If in the figure of Theorem 1 the $\angle$ **AEC** = $\frac{2}{3}$ $\angle$ **AED**, find the number of degrees in each $\angle$ of the figure.

4.

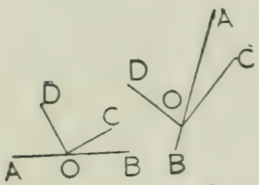


Fig. 1 Fig 2

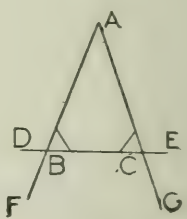
DOC is a rt. $\angle$, and through the vertex **O** a st. line **AOB** is drawn.

Prove that :—

In Fig. 1, $\angle$ **BOC** + $\angle$ **AOD** = a rt. $\angle$.
In Fig. 2, $\angle$ **BOC** − $\angle$ **AOD** = a rt. $\angle$.

5. In the diagram,

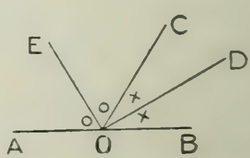
$\angle$ **ABC** = $\angle$ **ACB**.



Prove that

(1) $\angle$ **ABD** = $\angle$ **ACE**,
(2) $\angle$ **FBC** = $\angle$ **GCB**,
(3) $\angle$ **DBF** = $\angle$ **ECG**.

6. In the diagram,

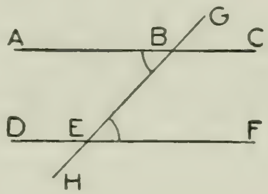


AOB is a st. line,
$\angle$ **COD** = $\angle$ **DOB** and
$\angle$ **AOE** = $\angle$ **EOC**.

Prove that **EOD** is a rt. $\angle$, and that $\angle$ **AOE** is the complement of $\angle$ **BOD**.

7. **E** is a point between **A** and **B** in the st. line **AB**; **DE**, **FE** are drawn on opposite sides of **AB** and such that $\angle$ **DEA** = $\angle$ **FEB**. Show that **DEF** is a st. line.

8. Four st. lines, OA, OB, OC, OD, are drawn in succession from the point O, and are such that ∠ AOB = ∠ COD and ∠ BOC = ∠ DOA. Show that AOC is a st. line, and also that BOD is a st. line.



9. In the diagram, ABC, DEF, GBEH are st. lines and ∠ ABE = ∠ BEF.

Prove that

(1) ∠ CBE = ∠ BED,

(2) ∠ GBC = ∠ DEH,

(3) ∠ ABG = ∠ BED,

(4) ∠s CBE, BEF are supplementary,

(5) ∠s ABE, BED are supplementary.

30. **Definitions.**—A figure formed by straight lines is called a **rectilineal** figure.

The figure formed by three straight lines which intersect one another is called a **triangle.**

The three points of intersection are called the **vertices** of the triangle.

The lines between the vertices of the triangle are called the **sides** of the triangle.

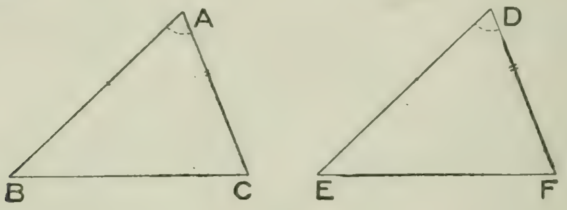
31. Figures that are equal in all respects, so that one may be made to fit the other exactly, are said to be **congruent.**

The sign ≡ is used to denote the congruence of figures.

First Case of the Congruence of Triangles

Theorem 2

If two triangles have two sides and the contained angle of one respectively equal to two sides and the contained angle of the other, the two triangles are congruent.



Hypothesis.—ABC and DEF are two $\triangle$s having AB = DE, AC = DF and $\angle$ A = $\angle$ D.

To prove that (1) BC = EF,

(2) $\angle$ B = $\angle$ E,

(3) $\angle$ C = $\angle$ F,

(4) area of $\triangle$ ABC = area of $\triangle$ DEF;

and, hence, $\triangle$ ABC $\equiv$ $\triangle$ DEF.

Proof.—Let $\triangle$ ABC be applied to $\triangle$ DEF so that vertex A falls on vertex D and AB falls along DE.

$\because$ AB = DE,

$\therefore$ vertex B must fall on vertex E.

$\because$ $\angle$ A = $\angle$ D,

$\therefore$ AC must fall along DF,

and $\therefore$, as AC = DF,

the vertex C must fall on the vertex F.

$\therefore$ $\triangle$ ABC coincides with $\triangle$ DEF.

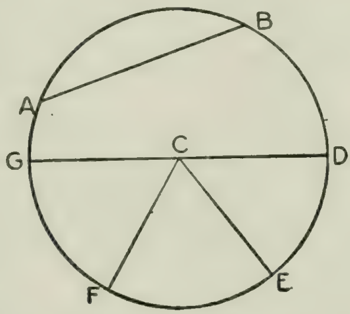
and $\therefore$ $\triangle$ ABC $\equiv$ $\triangle$ DEF.

32. **Definitions.**—A closed figure formed by four straight lines is called a **quadrilateral.**

In a quadrilateral a straight line joining two opposite vertices is called a **diagonal.**

A quadrilateral having its four sides equal to each other is called a **rhombus.**

A **circle** is a figure consisting of one closed curved line, called the **circumference,** and is such that all straight lines drawn from a certain point within the figure, called the **centre,** to the circumference are equal to each other.



In a circle a st. line drawn from the centre to the circumference is called a **radius.** (Plural—radii.)

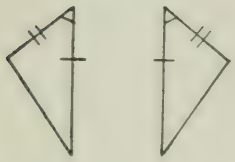
A st. line, as AB, joining two points in the circumference is called a **chord.**

If a chord passes through the centre, as GD, it is called a **diameter.**

A part of the circumference, as the curved line FED, is called an **arc.**

A line drawn from a point in one arm of an angle to a point in the other arm is said to **subtend** the angle. In the diagram the arc FE subtends the $\angle$ FCE; or in any $\triangle$ each side subtends the opposite $\angle$.

33.—Exercises



1. Prove Theorem 2 when one △ has to be supposed to be turned over before it can be made to coincide with the other.

2. The ∠ **B** of a △ **ABC** is a rt. ∠, and **CB** is produced to **D** making **BD** = **BC**. Prove **AD** = **AC**.

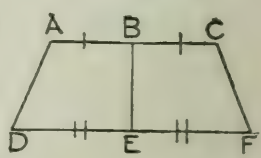
3. **A, B, C** are three points in a st. line such that **AB** = **BC**. **DB** is ⊥ **AC**. Show that any point in **DB**, produced in either direction, is equidistant from **A** and **C**.

4. Two st. lines **AOB, COD** cut one another at **O**, so that **OA** = **OB** and **OC** = **OD**; join **AD** and **BC**, and prove △s **AOD, BOC** congruent.

5. Prove that all chords of a circle which subtend equal angles at the centre are equal to each other.

6. If with the same centre **O**, two circles be drawn, and st. lines **ODB, OEC** be drawn to meet the circumferences in **D, E, B, C**; prove that **BE** = **DC**.

7. **ABCD** is a quadrilateral having the opposite sides **AB, CD** equal and ∠ **B** = ∠ **C**. Show that **AC** = **BD**.



8. In the diagram, **ABC** and **DEF** are both ⊥ **BE**. Also **AB** = **BC** and **DE** = **EF**. Prove that **AD** = **CF**.

9. Two st. lines **AOB, COD** cut one another at rt. ∠s at **O**. **AO** is cut off = **OB**, and **CO** = **OD**. Prove that the quadrilateral **ACBD** is a rhombus.

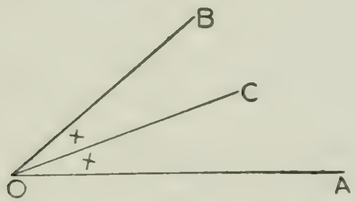
10. Two quadrilaterals **ABCD, EFGH** have **AB** = **EF**, **BC** = **FG**, **CD** = **GH**, ∠ **B** = ∠ **F**, ∠ **C** = ∠ **G**. Prove that they are congruent.

34. **Definitions.**—A triangle having its sides all equal to each other is called an **equilateral** triangle.

A triangle having two sides equal to each other is called an **isosceles** triangle.

A triangle having no two of its sides equal to each other is called a **scalene** triangle.

35.



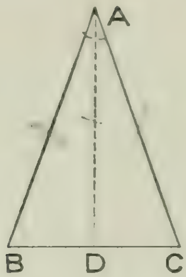
If a straight line revolve in the positive direction about the point O from the position OA to the position OB, it must pass through some position OC such that $\angle$ AOC = $\angle$ COB.

A straight line which divides an angle into two equal angles is called the **bisector** of the angle.

When a construction is represented in a diagram, although it has not previously been proved that it can be made, it is called a **hypothetical construction.** Thus OC has been drawn to represent the bisector of $\angle$ AOB.

THEOREM 3

The angles at the base of an isosceles triangle are equal to each other.



Hypothesis.—ABC is an isosceles $\triangle$ having AB = AC.

To prove that $\angle$ B = $\angle$ C.

Hypothetical Construction.—Draw the st. line AD to represent the bisector of $\angle$ BAC.

Proof.—In the two $\triangle$s ADB, ADC,

$$\left\{\begin{array}{ll} \text{AB} = \text{AC}, & (\textit{Hyp.}) \\ \text{AD is common}, & \\ \angle \text{BAD} = \angle \text{CAD}, & (\textit{Const.}) \end{array}\right.$$

$\therefore \triangle \text{ADB} \equiv \triangle \text{ADC},$ (I—2, page 16.)

$\therefore \angle \text{B} = \angle \text{C}.$

---

36. The two $\triangle$s ADB, ADC, in the diagram of Theorem 3, are congruent, and if the isosceles $\triangle$ be folded along the bisector of the vertical $\angle$ as crease, the parts on one side of the bisector will exactly fit the corresponding parts on the other side.

Definition.—When a figure can be folded along a line so that the part on one side exactly fits the part on the other side, the figure is said to be **symmetrical** with respect to that line.

The line along which the figure is folded is called an **axis of symmetry** of the figure.

Hence the bisector of the vertical ∠ of an isosceles △ is an axis of symmetry of the △.

It follows from the above definition of a symmetrical figure that—

If a figure is symmetrical with respect to a st. line, for every point on one side of this axis of symmetry there is a corresponding point on the other side.

Show by folding, in the diagram of Theorem 3, that if ∠ B = ∠ C, the side AB = the side AC.

37.—Exercises

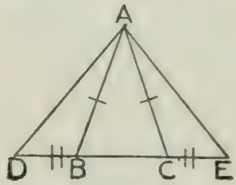
1. An equilateral △ is equiangular.

2. ABC is an equilateral △, and points D, E, F, are taken in BC, CA, AB respectively, such that BD = CE = AF. Show that DEF is an equilateral △.

3. Show that the exterior ∠s at the base of an isosceles △ are equal to each other.

4. The opposite ∠s of a rhombus are equal to each other.

5. ABC is an isosceles △ having AB = AC, and the base BC produced to D and E such that BD = CE. Prove that ADE is an isosceles △.



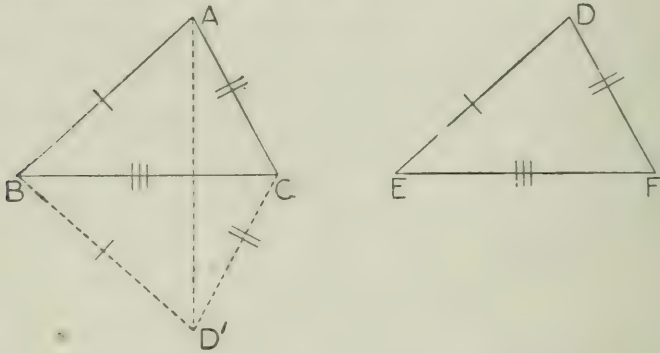
6. AC, AD are two st. lines on opposite sides of AB. Prove that if the bisectors of ∠s BAC, BAD are at rt. ∠s, AC, AD must be in the same st. line.

7. If a figure be symmetrical with respect to a st. line, the st. line joining any two corresponding points cuts the axis at rt. ∠s.

Second Case of the Congruence of Triangles

Theorem 4

If two triangles have the three sides of one respectively equal to the three sides of the other, the two triangles are congruent.



Hypothesis.—ABC, DEF are two △s having AB = DE, AC = DF and BC = EF.

To prove that △ ABC ≡ △ DEF.

Proof.—Let △ DEF be applied to △ ABC so that the vertex E falls on the vertex B and EF falls along BC.

Then ∵ EF = BC, the vertex F falls on C. Let D take the position D′ on the side of BC remote from A.

Join AD′.

$$\because \quad BA = BD',$$

$$\therefore \quad \angle BAD' = \angle BD'A. \qquad \text{(I—3, p. 20.)}$$

$$\text{Similarly} \quad \angle CAD' = \angle CD'A.$$

$$\therefore \angle BAD' + \angle CAD' = \angle BD'A + \angle CD'A,$$

$$\textit{i.e.,} \quad \angle BAC = \angle BD'C.$$

$$\text{Then in }\triangle\text{s BAC, BD'C} \begin{cases} BA = BD', \\ CA = CD', \\ \angle BAC = \angle BD'C, \end{cases}$$

$$\therefore \quad \triangle ABC \equiv \triangle BD'C; \qquad \text{(I—2, p, 16.)}$$

$$\textit{i.e.,} \quad \triangle ABC \equiv \triangle DEF.$$

Note.—In the proof of this theorem three cases may occur :—AD′ may cut BC as in Fig. 1, or not cut BC as in Fig. 2, or pass through one end of BC as in Fig. 3.

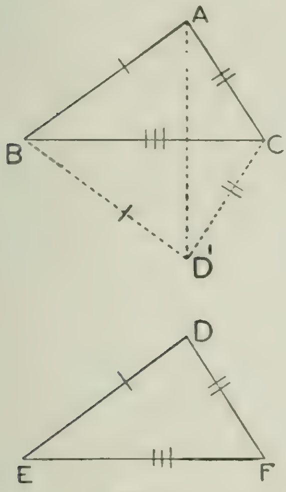


FIG. 1

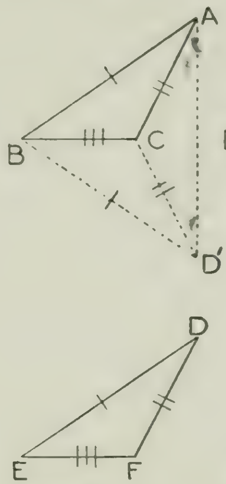


FIG. 2

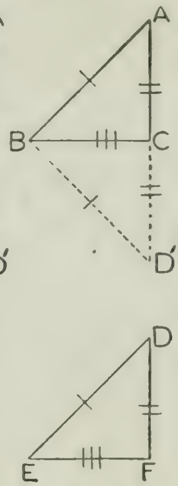


FIG. 3

The proof given above is that of the first case. The pupil should work out the proofs of the other two cases.

38.—Exercises

1. If the opposite sides of a quadrilateral be equal, the opposite $\angle$s are equal.

2. A diagonal of a rhombus bisects each of the $\angle$s through which it passes, and consequently, the diagonal is an axis of symmetry in the rhombus.

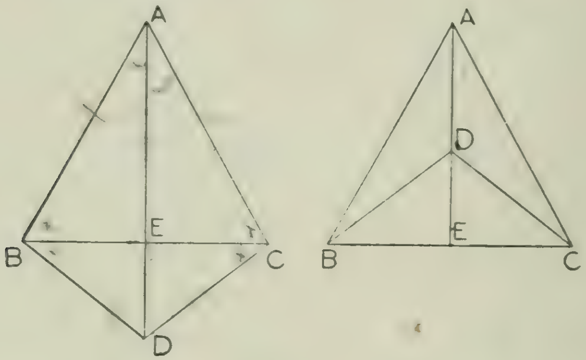
3. If in a quadrilateral ABCD the sides AB, CD be equal and $\angle$ ABC = $\angle$ BCD, prove that $\angle$ CDA = $\angle$ DAB.

4. Show that equal chords in a circle subtend equal $\angle$s at the centre.

5. Prove that the diagonals of a rhombus bisect each other at rt. $\angle$s.

Theorem 5

If two isosceles triangles are on the same base, the straight line joining their vertices is an axis of symmetry of the figure; and the ends of the base are corresponding points.



Hypothesis.—ABC, DBC are two isosceles $\triangle$s on the same base BC.

To prove that AD is an axis of symmetry of the figure.

Proof.—AD, or AD produced, cuts BC at E.

$$\text{In } \triangle\text{s ABD, ACD,} \begin{cases} \text{AB} = \text{AC} \\ \text{BD} = \text{CD}, \\ \text{AD is common,} \end{cases}$$

$$\therefore \triangle \text{BAD} \equiv \triangle \text{CAD}. \qquad \text{(I—4, p. 22.)}$$

$$\text{and } \therefore \angle \text{BAD} = \angle \text{CAD}.$$

$$\text{In } \triangle\text{s BAE, CAE,} \begin{cases} \text{BA} = \text{CA}, \\ \text{AE is common,} \\ \angle \text{BAE} = \angle \text{CAE}, \end{cases}$$

$$\therefore \triangle \text{BAE} \equiv \triangle \text{CAE}. \qquad \text{(I—2, p. 16.)}$$

$$\text{Similarly, } \triangle \text{BDE} \equiv \triangle \text{CDE}.$$

Hence, each part of the figure on one side of **AD** is congruent to the corresponding part on the other side, and if the figure be folded on **AD**, as crease, the corresponding parts will coincide.

$\therefore$ **AD** is an axis of symmetry of the figure; and **B**, **C** are corresponding points.

39.—**Exercises**

1. If two circles cut at two points, the st. line which joins their centres bisects at rt. $\angle$s the st. line joining the points of section.

2. **A**, **B**, **C** are three points each of which is equidistant from two fixed points **P**, **Q**. Show that **A**, **B**, **C** are in a st. line which bisects the st. line joining **P**, **Q** and cuts it at rt. $\angle$s.

Constructions

40. In Theoretical Geometry the use of instruments in making constructions is generally restricted to an ungraduated straight edge and a pair of compasses. With these instruments we can:—

1. Draw a st. line from one point to another.

2. Produce a st. line.

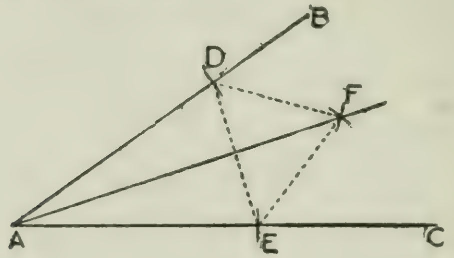
3. Describe a circle with any point as its centre and radius equal to any given st. line.

4. Cut off from one st. line a part equal to another st. line.

Note.—*All constructions should be accurately and neatly drawn by the pupil, and, by means of theorems already proved, the correctness of the method of construction should be shown.*

Problem 1

To bisect a given angle.



Let **BAC** be the given $\angle$.

Construction.—With the compasses cut off equal distances **AD** and **AE** from the arms of the $\angle$.

With centre **D** describe an arc.

With centre **E** and the same radius describe another arc cutting the first at **F**.

Join **AF**.

Then **AF** is the bisector of $\angle$ **BAC**.

Proof.—Join **DF**, **EF**, **DE**.

ADE, **FDE** are isosceles $\triangle$s on the same base **DE**,

$\therefore$ **AF** is an axis of symmetry of the figure, (I—5, p. 24.)

$\therefore$ **AF** bisects $\angle$ **BAC**.

Note.—*The equal radii for the arcs with centres* **D** *and* **E** *must be taken long enough for the arcs to intersect.*

41.—Exercises

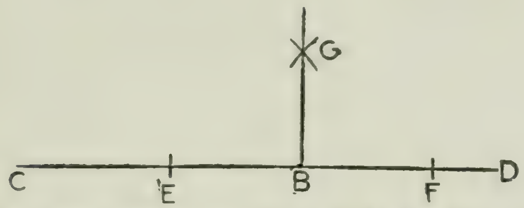
1. Divide a given $\angle$ into four equal parts.

2. Prove that the bisectors of a pair of vertically opposite $\angle$s are in the same st. line.

3. Bisect a st. $\angle$.

PROBLEM 2

To draw a perpendicular to a given straight line from a given point in the line.



Let **CD** be the given st. line and **B** the given point.

Construction.—Bisect the st. ∠ **CBD** by the st. line **BG.**

Proof.—Then each of the ∠s **CBG, DBG** is half of a st. ∠ and ∴ each is a rt. ∠.

∴ **BG** is ⊥ **CD.**

42.—**Exercises**

Using ruler and compasses only, construct ∠s of (1), 45°; (2), $22\frac{1}{2}$°; (3), 135°; (4), $67\frac{1}{2}$°; (5), 225°.

43. **Definitions.**—If one angle of a triangle be a right angle, the triangle is called a **right-angled** triangle.

In a right-angled triangle the side opposite the right angle is called the **hypotenuse.**

If one angle of a triangle be an obtuse angle, the triangle is called an **obtuse-angled** triangle.

If all three angles of a triangle be acute angles, the triangle is called an **acute-angled** triangle.

The **altitude** of a triangle is the length of the perpendicular from any vertex to the opposite side.

44.—Exercises

1. Construct a rt.-$\angle$d $\triangle$ having one of the arms of the rt. $\angle$ three times the other.

2. Construct a rt.-$\angle$d $\triangle$ having the hypotenuse three times one of the arms of the rt. $\angle$.

3. Given the length of the hypotenuse and of one of the sides of a rt.-$\angle$d $\triangle$, construct the $\triangle$.

4. Construct a rhombus having each of its diagonals equal to twice a given st. line.

5. Construct a rhombus having one diagonal twice and the other four times a given st. line.

6. Construct an isosceles $\triangle$ having given its altitude and the length of one of the equal sides.

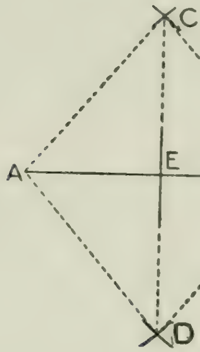
7. Construct an isosceles rt.-$\angle$d $\triangle$.

45. **Definitions.**—Sometimes when a proposition has been proved the truth of another proposition follows as an immediate consequence of the former; such a proposition is called a **corollary.**

A straight line which bisects a line of given length at right angles is called the **right bisector** of the line.

PROBLEM 3

To bisect a given straight line.



Let AB be the given st. line.

Construction.—With centre A and any radius that is plainly greater than half of AB, draw two arcs, one on each side of AB.

With centre B and the same radius draw two arcs cutting the first two at C and D.

Join CD, cutting AB at E.

E is the middle point of AB.

Proof.—Join CA, AD, DB, BC.

CAB, DAB are isosceles △s on the same base AB,

∴ CD is an axis of symmetry of the figure; and A, B are corresponding points. (I—5, p. 24.)

∴ AE = EB.

Corollary.—From the above proof it follows that **the ∠s at E are rt. ∠s,** and hence, **CD is the right bisector of AB.**

46. **Definition.**—The straight line drawn from a vertex of a triangle to the middle point of the opposite side is called a **median** of the triangle.

47.—Exercises

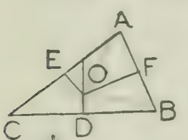
1. Divide a given st. line into four equal parts.

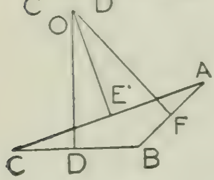
2. In an isosceles $\triangle$ prove that the bisector of the vertical $\angle$ is a median of the $\triangle$.

3. In an equilateral $\triangle$ prove that the bisectors of the $\angle$s are medians of the $\triangle$.

4. Show that any point in the right bisector of a given st. line is equidistant from the ends of the given line.

5. In any $\triangle$ the point of intersection of the right bisectors of any two sides is equidistant from the three vertices.





6. The right bisectors of the three sides of a $\triangle$ pass through one point.

The right bisectors of AB, BC *meet at* O. *Bisect* AC *at* E. *Join* EO. *Prove* OE $\perp$ AC.

7. Describe a circle through the three vertices of a $\triangle$.

8. Describe a circle to pass through three given points that are not in the same st. line.

9. Show how any number of circles may be drawn through two given points.

What line contains the centres of all these circles?

10. In a given st. line find a point that is equally distant from two given points.

11. On a given base describe an isosceles $\triangle$ so that the sum of the two equal sides may equal a given st. line.

In what case is this impossible?

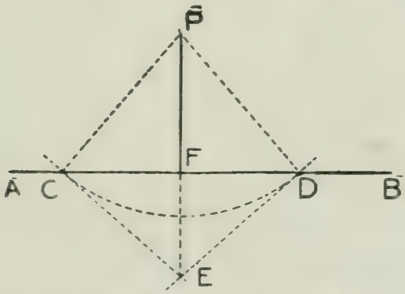
12. Construct a rhombus having its diagonals equal to two given st. lines.

13. In $\triangle$ ABC find in CA, produced if necessary, a point D so that DC = DB.

14. In $\triangle$s ABC, DEF, AB = DE, AC = DF and the medians drawn from B and E are equal to each other. Prove that $\triangle$ ABC $\equiv \triangle$ DEF.

Problem 4

To draw a perpendicular to a given straight line from a given point without the line.



Let P be the given point and AB the given st. line.

Construction.—Describe an arc with centre P to cut AB at C and D.

With centres C and D, and equal radii, describe two arcs cutting at E.

Join PE, cutting AB at F.

PF is the required perpendicular.

Proof.—Join PC, CE, ED, DP.

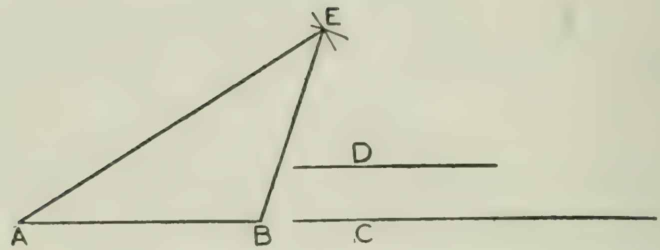
$\because$ PCD, ECD are isosceles $\triangle$s on the same base CD,

$\therefore$ PE is an axis of symmetry of the figure; and C, D are corresponding points. (I—5, p. 24.)

$\therefore$ $\angle$s at F are rt. $\angle$s, and PF is $\perp$ AB.

Problem 5

To construct a triangle with sides of given length.



Let AB, C and D be the given lengths.

Construction.—With centre A and radius C describe an arc.

With centre B and radius D describe an arc cutting the first arc at E.

Join EA, EB.

AEB is the required △.

Question—*In what case would the above construction fail?*

48.—Exercises

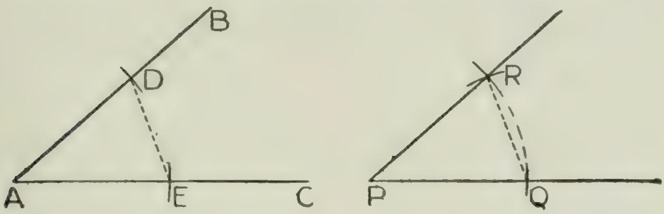
1. On a given st. line describe an equilateral △.

2. On a given base describe an isosceles △ having each of the equal sides double the base.

3. Construct a rhombus having given a diagonal and the length of one of the equal sides.

Problem 6

To construct an angle equal to a given angle.



Let BAC be the given $\angle$.

Construction. — From AC, AB cut off equal parts AE, AD.

Draw a line and mark a point P in it.

Cut off PQ = AE.

With centre P and radius PQ describe an arc.

With centre Q and radius DE describe an arc cutting the arc with centre P at R.

Join RP.

RPQ is the required $\angle$.

Proof.—Join DE, RQ.

In $\triangle$s PRQ, ADE. $\left\{\begin{array}{l} PQ = AE, \\ PR = AD, \\ RQ = DE, \end{array}\right.$

$\therefore \angle RPQ = \angle BAC.$ (I—4, p. 22.)

49.—Exercises

1. Construct a rhombus having given one of its ∠s and the length of one of its equal sides.

2. Construct a quadrilateral equal in all respects to a given quadrilateral.

3. On a given st. line **BC** construct a △ having the ∠s **B, C** equal to two given acute ∠s.

4. Construct an ∠ equal to the complement of a given acute ∠.

5. Construct an ∠ equal to the supplement of a given ∠.

6. On a given base describe an isosceles △ having its altitude equal to a given st. line.

7. In the side **BC** of a △ **ABC** find a point **E**, such that **AE** is half the sum of **AB** and **AC**.

8. The △ formed by joining the middle points of the three sides of an isosceles △ is isosceles.

9. **AB** is a given st. line and **C** is a given point without the line. Find the point **D** so that **C** and **D** may be symmetrical with respect to **AB**.

10. **C, D** are given points, (1) on opposite sides, (2) on the same side of a given st. line **AB**. Find a point **P** in **AB** so that **CP, DP** make equal ∠s with **AB**.

11. The right bisectors of the two sides **AB, AC** of △ **ABC** meet at **D**, and **E** is the middle point of **BC**. Show that **DE** ⊥ **BC**.

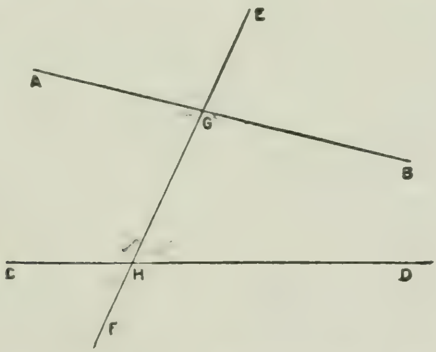
Parallel Straight Lines

50. **Definitions.**—Two straight lines *in the same plane* which do not meet when produced for any finite distance in either direction are said to be **parallel** to each other.

A straight line which cuts two, or more, other straight lines is called a **transversal.**

A quadrilateral that has both pairs of opposite sides parallel to each other is called a **parallelogram.**

Draw a st. line EF cutting two other st. lines AB and CD at G and H.



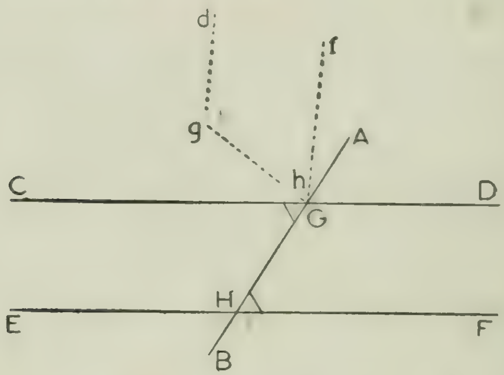
Eight ∠s are thus formed, four of which, AGH, BGH, CHG, DHG, being between AB and CD, are called **interior** ∠s. The other four are called **exterior** ∠s.

The interior ∠s AGH and GHD, on opposite sides of the transversal, are called **alternate** ∠s. Thus also, BGH and GHC are alternate ∠s.

Name four pairs of equal angles in the diagram.

Theorem 6

If a transversal meeting two straight lines makes the alternate angles equal to each other, the two straight lines are parallel.



Hypothesis. — The transversal AB meeting CD and EF makes ∠ CGH = the alternate ∠ GHF.

To prove that CD ∥ EF.

Proof.—Detach the part DGHF from the figure and mark it $d\,g\,h\,f$.

Slide $d\,g\,h\,f$, from its original position, along the transversal until h comes to the point G.

Then, rotate $d\,g\,h\,f$, in either direction, through a st. ∠ about the point G.

When the rotation is complete $h\,g$ coincides with G H.

And, ∵ ∠ $f\,h\,g$ = ∠ CGH,

∴ $h\,f$ coincides with GC.

Also, ∵ ∠ $d\,g\,h$ = ∠ GHE,

∴ $g\,d$ coincides with HE.

If it be possible let **CD** and **EF** when produced meet towards **D** and **F**.

Then $h\ f$ and $g\ d$ must meet towards f and d,

$\therefore$ **GC** and **HE** must meet towards **C** and **E**.

Hence, **CD** and **EF** when produced must meet in two points.

This is impossible by the definition of a st. line.

$\therefore$ **CD** and **EF** do not meet towards **D** and **F**, and hence cannot meet towards **C** and **E**.

$\therefore$ **CD** $\parallel$ **EF**.

NOTE.—*If this proof is not at once clear to the pupil he should make a drawing of the diagram, cut out the part d g h f, and turning it about, fit it to* **E H G C**.

51.—Exercises

1. Lines which are $\perp$ to the same st. line are $\parallel$ to each other.

2. If both pairs of opposite sides of a quadrilateral are equal to each other, the quadrilateral is a $\parallel$gm.

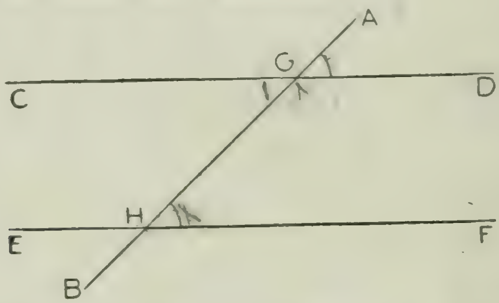
3. A rhombus is a $\parallel$gm.

4. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a $\parallel$gm.

5. No two st. lines drawn from two vertices of a $\triangle$, and terminated in the opposite sides, can bisect each other.

Theorem 7

If a transversal meeting two straight lines makes (1) an exterior angle equal to the interior and opposite angle on the same side of the transversal, or, (2) the two interior angles on the same side of the transversal supplementary, in either case the two straight lines are parallel.



(1) *Hypothesis.*— AB meeting CD, EF makes ∠ AGD = ∠ GHF.

To prove.— CD ∥ EF.

Proof.— ∠ CGH = ∠ AGD, (I—1, p. 13.)

but ∠ AGD = ∠ GHF, (*Hyp.*)

∴ ∠ CGH = ∠ GHF.

∴ CD ∥ EF. (I—6, p. 36.)

(2) *Hypothesis.*—AB meeting CD, EF makes ∠ DGH + ∠ GHF = two rt. ∠s.

To prove.—CD ∥ EF.

Proof.—∠ CGH + ∠ DGH = two rt. ∠s,

but ∠ DGH + ∠ GHF = two rt. ∠s, (*Hyp.*)

∴ ∠ CGH + ∠ DGH = ∠ DGH + ∠ GHF.

From each take the common ∠ DGH, and ∠ CGH = ∠ GHF,

∴ CD ∥ EF (I—6, p. 36.)

52. The following statement of a fundamental property of parallel straight lines is called **Playfair's axiom**:—

Through any point one, and only one, straight line can be drawn parallel to a given straight line.

From this axiom it follows that:—

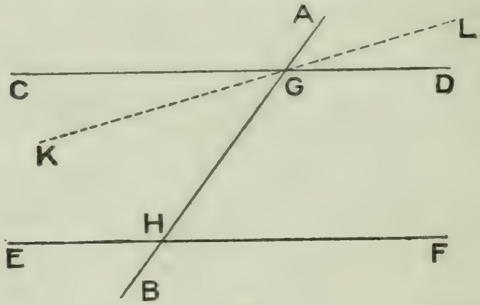
No two intersecting straight lines can be parallel to the same straight line.

∴ straight lines which are parallel to the same straight line are not intersecting lines, *i.e.* :—

Straight lines which are parallel to the same straight line are parallel to each other.

Theorem 8

If a transversal cuts two parallel straight lines, the alternate angles are equal to each other.



Hypothesis.—The transversal AB cuts the || st. lines CD, EF at G, H.

To prove that ∠ CGH = ∠ GHF.

Proof.—If ∠ CGH be not equal to ∠ GHF, make the ∠ KGH = ∠ GHF, and produce KG to L.

Then ∵ AB cuts KL and EF, making ∠ KGH = the alternate ∠ GHF.

∴ KL is || to EF. (I—6, p. 36.)

But CD is, by hypothesis, || to EF.

That is, two intersecting st. lines, KL and CD, are both || EF, which is impossible.

∴ ∠ CGH = ∠ GHF.

53. Consider the method of proof used in Theorem 8.

To prove that ∠ CGH = ∠ GHF we began by assuming that these ∠s are not equal, and then showed that something absurd or contrary to the hypothesis must follow, and concluded that ∠ CGH = ∠ GHF.

This method of proof, in which we begin by assuming that the conclusion is not true, is called the **indirect method of demonstration.**

54. Compare Theorems 6 and 8.

In both cases a transversal cuts two straight lines.

In Theorem 6 the hypothesis is that the alternate angles are equal, and the conclusion is that the lines are parallel.

In Theorem 8 the hypothesis is that the lines are parallel, and the conclusion is that the alternate angles are equal.

Thus in these propositions the hypothesis of each is the conclusion of the other.

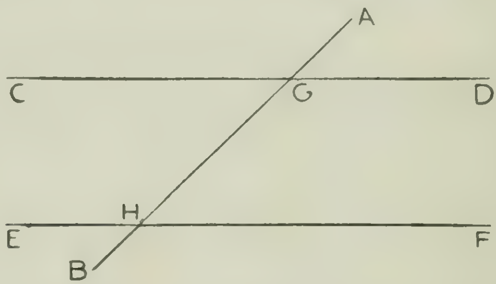
When two propositions are such that the hypothesis of each is the conclusion of the other, they are said to be **converse propositions**; or each is said to be the converse of the other.

The converse of a true proposition may, or may not, be true. The converse propositions in Theorems 6 and 8 are both true; but consider the true proposition:— All rt. $\angle$s are equal to each other; and its converse:— All equal $\angle$s are rt. $\angle$s. The last is easily seen to be untrue. Consequently proof must in general be given for each of a pair of converse propositions.

When a proposition is known to be true and we wish to prove the converse we commonly use the indirect method.

Theorem 9

If a transversal cuts two parallel straight lines, it makes (1) an exterior angle equal to the interior and opposite angle on the same side of the transversal, and (2) the interior angles on the same side of the transversal supplementary.



Hypothesis.—AB cuts the || st. lines CD, EF.

To prove that (1) $\angle$ AGD = $\angle$ AHF.

(2) $\angle$ DGH + $\angle$ GHF = two rt. $\angle$s.

Proof.—(1) $\because$ CD || EF,

$\therefore$ $\angle$ GHF = $\angle$ CGH. (I—8, p. 40.)

but $\angle$ CGH = $\angle$ AGD, (I—1, p. 13.)

$\therefore$ $\angle$ AGD = $\angle$ GHF.

(2) $\because$ $\angle$ GHF = $\angle$ CGH,

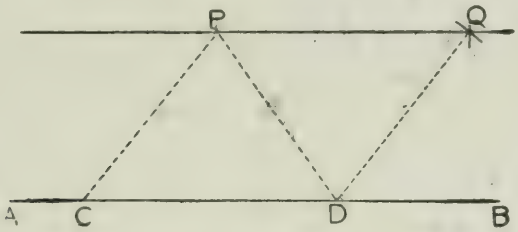
$\therefore$ $\angle$ GHF + $\angle$ DGH = $\angle$ CGH + $\angle$ DGH;

but $\angle$ CGH + $\angle$ DGH = a st. $\angle$

$\therefore$ $\angle$s GHF, DGH are supplementary.

Problem 7

Through a given point to draw a straight line parallel to a given straight line.



Let P be the given point and AB the given st. line.

Construction.—Take two points C, D, in AB.

With centre P and radius CD describe an arc.

With centre D and radius CP describe an arc cutting the first at Q.

Join PQ.

Then PQ $\parallel$ AB.

Proof.—Join PC, DQ, PD.

In $\triangle$s PCD, DQP, $\begin{cases} PC = DQ, \\ CD = QP, \\ PD \text{ is common,} \end{cases}$

$\therefore \angle CDP = \angle DPQ.$ (I—4, p. 22.)

$\therefore$ PQ $\parallel$ AB. (I—6, p. 36.)

55.—Exercises

1. If a st. line be $\perp$ to one of two $\parallel$ st. lines, it is also $\perp$ to the other.

2. Prove, by using a transversal, that st. lines which are $\parallel$ to the same st. line are $\parallel$ to each other.

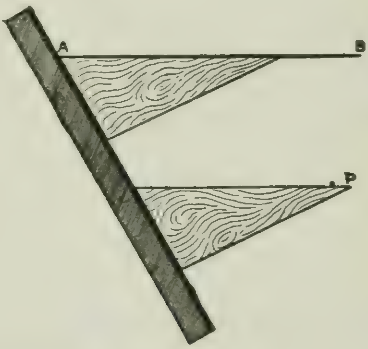
3. Any st. line || to the base of an isosceles △ makes equal ∠s with the sides, or the sides produced.

4. Construct a △ having two of its ∠s respectively equal to two given ∠s, and the length of the ⊥ from the vertex of the third ∠ to the opposite side equal to a given st. line.

5. Construct a rt.-∠d △ having given one side and the opposite ∠.

6. If one ∠ of a ||gm be a rt. ∠, the other three ∠s are also rt. ∠s.

7. Give a proof for the following method of drawing a line through **P** || **AB** :—



Place the set-square with the hypotenuse along the st. line **AB**.

Place a ruler against another side of the set-square as in the diagram.

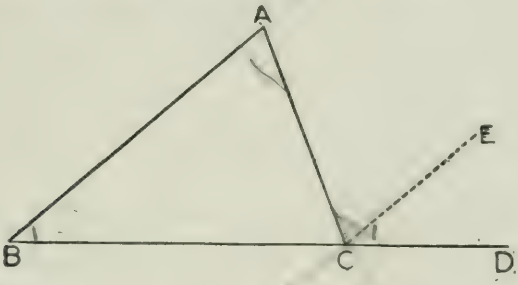
Hold the ruler firmly in position and slide the set-square along it until the hypotenuse comes to the point **P**.

A line drawn through **P** along the set-square is || **AB**.

TRIANGLES

THEOREM 10

The exterior angle, made by producing one side of a triangle, equals the sum of the two interior and opposite angles; and the three interior angles are together equal to two right angles.



Hypothesis.—ABC is a $\triangle$ having BC produced to D.

To prove that (1) $\angle ACD = \angle A + \angle B$.

(2) $\angle A + \angle B + \angle ACB =$ two rt. $\angle$s.

Construction.—Through C draw CE $\parallel$ AB.

Proof.— $\because$ CE $\parallel$ AB,

and AC is a transversal,

$\therefore \angle ACE = \angle A$. (I—8, p. 40.)

$\because$ BD is a transversal,

$\therefore \angle ECD = \angle B$. (I—9, p. 42.)

$\therefore \angle ACE + \angle ECD = \angle A + \angle B$.

i.e., $\angle ACD = \angle A + \angle B$.

Hence, $\angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$.

But $\angle ACD + \angle ACB =$ two rt. $\angle$s,

$\therefore \angle A + \angle B + \angle ACB =$ two rt. $\angle$s.

Cor.—The exterior angle of a triangle is greater than either of the interior and opposite angles.

56.—Exercises

1. Prove Theorem 10 by means of a st. line drawn through the vertex || the base.

2. **If two $\triangle$s have two $\angle$s of one respectively equal to two $\angle$s of the other, the third $\angle$ of one is equal to the third $\angle$ of the other.**

3. **The sum of the $\angle$s of a quadrilateral is equal to four rt. $\angle$s.**

4. The sum of the $\angle$s of a pentagon is six rt. $\angle$s.

5. Each $\angle$ of a equilateral $\triangle$ is an $\angle$ of $60°$.

6. Find a point B in a given st. line CD such that, if AB be drawn to B from a given point A, the $\angle$ ABC will equal a given $\angle$.

7. Show that the bisectors of the two acute $\angle$s of a rt.-$\angle$d $\triangle$ contain an $\angle$ of $135°$.

8. If both pairs of opposite $\angle$s of a quadrilateral are equal, the quadrilateral is a ||gm.

9. C is the middle point of the st. line AB. CD is drawn in any direction and equal to CA or CB. Prove that ADB is a rt. $\angle$.

10. On AB, AC, sides of a $\triangle$ ABC, equilateral $\triangle$s ABD, ACE are described externally. Show that DC = BE.

11. AB is any chord of a circle of which the centre is O. AB is produced to C so that BC = BO. CO is joined, cutting the circle at D and is produced to cut it again at E. Show that $\angle$ AOE = three times $\angle$ BCD.

12. If the exterior $\angle$s at B and C of a $\triangle$ ABC be bisected and the bisectors be produced to meet at D, the $\angle$ BDC equals half the sum of $\angle$s ABC, ACB.

13. **Show that a △ must have at least two acute ∠s.**

14. In an acute-∠d △ show that the ⊥ from a vertex to the opposite side cannot fall outside of the △.

15. In an obtuse-∠d △ show that the ⊥ from the vertex of the obtuse ∠ on the opposite side falls within the △, but that the ⊥ from the vertex of either acute ∠ on the opposite side falls outside of the △.

16. In a rt.-∠d △ where do the ⊥s from the vertices on the opposite sides fall?

17. Only one ⊥ can be drawn from a given point to a given st. line.

18. Not more than two st. lines each equal to the same given st. line can be drawn from a given point to a given st. line.

19. D is a point taken within the △ ABC. Join DB, DC; and show, by producing BD to meet AC, that ∠ BDC > ∠ BAC.

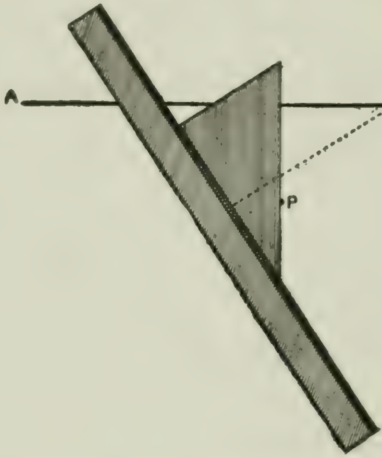
20. With compasses and ruler only, construct the following ∠s:—30°, 15°, 120°, 105°, 75°, $67\frac{1}{2}$°, 150°, 195°, 210°, 240°, 255°, 285°, − 30°, − 75°, − 135°.

21. If a transversal cut two st. lines so as to make the interior ∠s on one side of the transversal together less than two rt. ∠s, the two lines when produced shall meet on that side of the transversal.

22. The bisector of the exterior vertical ∠ of an isosceles △ is ∥ to the base.

23. Give a proof for the following method of drawing a line through P ⊥ AB:—

First place the set-square in the position shown by the dotted line, with its hypotenuse along **AB**.



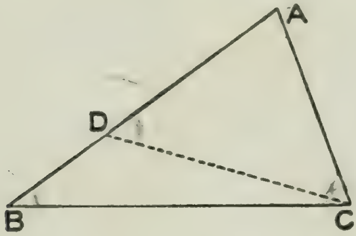
Place a ruler along one of the sides of the set-square and hold it firmly in that position.

Rotate the set-square through its right $\angle$, thus bringing the other side against the ruler, and slide the set-square along the ruler to the position shown by the shaded $\triangle$.

A line drawn through **P**, along the hypotenuse of the set-square, is perpendicular to **AB**.

Theorem 11

If one side of a triangle is greater than another side, the angle opposite the greater side is greater than the angle opposite the less side.



Hypothesis.—ABC is a $\triangle$ having AB $>$ AC.

To prove that $\angle$ ACB $>$ $\angle$ ABC.

Construction.—From AB cut off AD = AC. Join DC.

Proof.—In $\triangle$ ADC,

$\because$ AD = AC,

$\therefore$ $\angle$ ADC = $\angle$ ACD. (I—3, p. 20.)

But $\angle$ ACB $>$ $\angle$ ACD,

$\therefore$ $\angle$ ACB $>$ $\angle$ ADC.

In $\triangle$ BDC,

$\because$ BD is produced to A,

$\therefore$ exterior $\angle$ ADC $>$ interior and opposite $\angle$ DBC. (I—10, Cor., p. 45.)

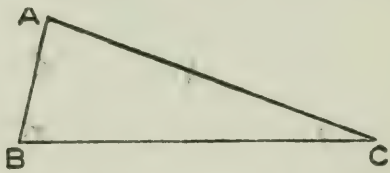
But $\angle$ ACB $>$ $\angle$ ADC;

much more $\therefore$ is $\angle$ ACB $>$ $\angle$ ABC.

Theorem 12

(Converse of Theorem 11)

If one angle of a triangle is greater than another angle of the same triangle, the side opposite the greater angle is greater than the side opposite the less.



Hypothesis.—In $\triangle$ ABC $\angle$ B $>$ $\angle$ C.

To show that AC $>$ AB.

Proof.—If AC be not $>$ AB,

then either AC $=$ AB,

or AC $<$ AB.

If AC $=$ AB,

then $\angle$ B $=$ $\angle$ C. (I—3, p. 20.)

But this is not so, $\therefore$ AC is not $=$ AB.

If AC $<$ AB,

then $\angle$ B $<$ $\angle$ C. (I—11, p. 49.)

But this also is not so, $\therefore$ AC is not $<$ AB.

Hence $\because$ AC is neither $=$ nor $<$ AB,

$\therefore$ AC $>$ AB.

57.—Exercises

1. **The perpendicular is the shortest st. line that can be drawn from a given point to a given straight line.**

The length of the $\perp$ from a given point to a given st. line is called the distance of the point from the line.

2. **ABCD** is a quadrilateral, of which **AD** is the longest side, and **BC** the shortest. Show that ∠ **B** > ∠ **D**, and that ∠ **C** > ∠ **A**.

3. The hypotenuse of a rt.-∠d △ is greater than either of the other two sides.

4. A st. line drawn from the vertex of an isosceles △ to any point in the base is less than either of the equal sides.

5. A st. line drawn from the vertex of an isosceles △ to any point in the base produced is greater than either of the equal sides.

6. If one side of a △ be less than another, the ∠ opposite the less side is acute.

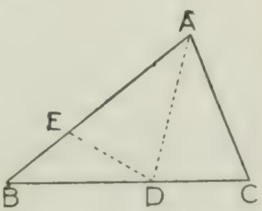
7. If **D** be any point in the side **BC** of a △ **ABC**, the greater of the sides **AB**, **AC**, is greater than **AD**.

8. **AB** is drawn from **A** ⊥ **CD**. **E**, **F** are two points in **CD** on the same side of **B**, and such that **BE** < **BF**. Show that **AE** < **AF**. Prove the same proposition when **E**, **F** are on opposite sides of **B**.

9. **ABC** is a △ having **AB** > **AC**. The bisector of ∠ **A** meets **BC** at **D**. Show that **BD** > **DC**. Give a general statement of this proposition.

10. **ABC** is a △ having **AB** > **AC**. If the bisectors of ∠s **B**, **C** meet at **D**, show that **BD** > **DC**.

11. Prove Theorem 11 from the following construction: Bisect ∠ **A** by **AD** which meets **BC** at **D**; from **AB** cut off **AE** = **AC**, and join **ED**.



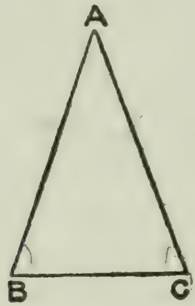
12. The ∠s at the ends of the greatest side of a △ are acute.

13. If **AB** > **AD** in the ||gm **ABCD**, ∠ **ADB** > ∠ **BDC**.

Theorem 13

(Converse of Theorem 3)

If two angles of a triangle are equal to each other, the sides opposite these equal angles are equal to each other.



Hypothesis.—In $\triangle$ ABC $\angle$ B = $\angle$ C.

To prove that AB = AC.

Proof.— If AB is not = AC,

let AB $>$ AC.

Then $\angle$ C $>$ $\angle$ B. (I—11, p. 49.)

But this is not so.

$\therefore$ AB is not $>$ AC.

Similarly it may be shown that

AB is not $<$ AC.

$\therefore$ AB = AC.

58.—Exercises

1. An equiangular $\triangle$ is equilateral.

2. BD, CD bisect the $\angle$ s ABC, ACB at the base of an isosceles $\triangle$ ABC. Show that $\triangle$ DBC is isosceles.

3. ABC is a $\triangle$ having AB, AC produced to D, E respectively. The exterior $\angle$ s DBC, ECB are bisected by

BF, **CF**, which meet at **F**. Show that, if **FB** = **FC**, the △ **ABC** is isosceles.

4. On the same side of **AB** the two △s **ACB**, **ADB** have **AC** = **BD**, **AD** = **BC**, and **AD**, **BC** meet at **E**. Show that **AE** = **BE**.

5. On a given base construct a △ having one of the ∠s at the base equal to a given ∠, and the sum of the sides equal to a given st. line.

6. On a given base construct a △ having one of the ∠s at the base equal to a given ∠ and the difference of the sides equal to a given st. line.

7. If the bisector of an exterior ∠ of a △ be ∥ to the opposite side, the △ is isosceles.

8. Through a point on the bisector of an ∠ a line is drawn ∥ to one of the arms. Prove that the △ thus formed is isosceles.

9. A st. line drawn ⊥ to **BC**, the base of an isosceles △ **ABC**, cuts **AB** at **X** and **CA** produced at **Y**. Show that **AXY** is an isosceles △.

10. **ACB** is a rt.-∠d △ having the rt. ∠ at **C**. Through **X**, the middle point of **AC**, **XY** is drawn ∥ **CB** cutting **AB** at **Y**. Show that **Y** is the middle point of **AB**.

11. **The middle point of the hypotenuse of a rt.-∠d △ is equidistant from the three vertices.**

12. **The st. line joining the middle points of two sides of a △ is ∥ to the third side.**

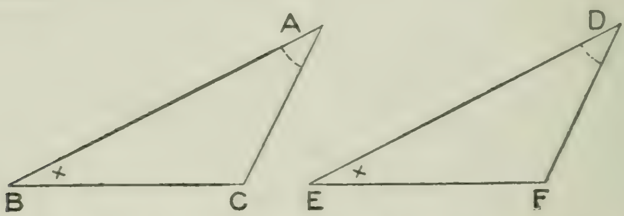
13. Construct a rt.-∠d △, having the hypotenuse equal to one given st. line, and the sum of the other two sides equal to another given st. line.

14. If one ∠ of a △ equals the sum of the other two, show that the △ is a rt.-∠d △.

Third Case of the Congruence of Triangles

Theorem 14

If two triangles have two angles and a side of one respectively equal to two angles and the corresponding side of the other, the triangles are congruent.



Hypothesis.—ABC, DEF are two $\triangle$s having $\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

To prove that $\triangle ABC \equiv \triangle DEF$.

Proof.— $\because \angle A = \angle D$,

and $\angle B = \angle E$,

$\therefore \angle A + \angle B = \angle D + \angle E$.

But $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$. (I—10, p. 45.)

$\therefore \angle C = \angle F$.

Apply $\triangle$ ABC to $\triangle$ DEF so that BC coincides with the equal side EF.

$\because \angle B = \angle E$,

$\therefore$ BA falls along ED, and A is on the line ED.

$\because \angle C = \angle F$,

$\therefore$ CA falls along FD, and A is on the line FD.

But D is the only point common to ED and FD,

$\therefore$ A falls on D.

$\therefore \triangle ABC$ coincides with $\triangle DEF$,

and $\therefore \triangle ABC \equiv \triangle DEF$.

59.—Exercises

1. If the bisector of an ∠ of a △ be ⊥ to the opposite side, the △ is isosceles.

2. Any point in the bisector of an ∠ is equidistant from the arms of the ∠.

3. In the base of a △ find a point that is equidistant from the two sides.

4. In a given st. line find a point that is equidistant from two other given st. lines.

5. Within a △ find a point that is equally distant from the three sides of the △.

6. Without a △ find three points each of which is equally distant from the three st. lines that form the △.

7. The ends of the base of an isosceles △ are equidistant from the opposite sides.

8. Two rt.-∠d △s are congruent, if the hypotenuse and an acute ∠ of one are respectively equal to the hypotenuse and an acute ∠ of the other.

9. Construct a △ with a side and two ∠s respectively equal to a given st. line and two given ∠s.

10. The ⊥ from the vertex of an isosceles △ to the base, bisects the base and the vertical ∠.

11. Prove I—13 by drawing the bisector of the vertical ∠, and using I—14.

12. △ ABC ≡ △ DEF and AX, DY are ⊥ to BC, EF respectively. Prove that AX = DY.

13. △ ABC ≡ △ DEF and AM, DN bisect ∠s A, D and meet BC, EF at M, N respectively. Prove that AM = DN.

14. If the diagonal AC of a quadrilateral ABCD bisects the ∠s at A and C, AC is an axis of symmetry of ABCD.

15. The middle point of the base of an isosceles △ is equidistant from the equal sides.

The Ambiguous Case in the Comparison of Triangles

Theorem 15

If two triangles have two sides of one respectively equal to two sides of the other and have the angles opposite one pair of equal sides equal to each other, the angles opposite the other pair of equal sides are either equal or supplementary.

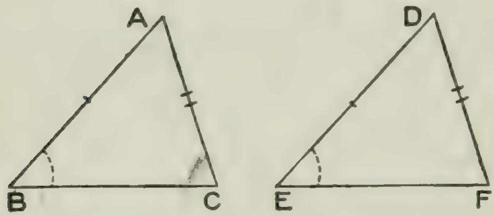


Fig. 1

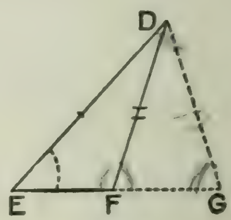


Fig. 2

Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and $\angle$ B = $\angle$ E.

To prove that either $\angle$C = $\angle$ F,

or $\angle$ C + $\angle$ F = two rt. $\angle$s.

Proof.—**Case I.** Suppose $\angle$ A = $\angle$ D. (Fig. 1.)

Then in the two $\triangle$s ABC, DEF,

$\because$ $\angle$ A = $\angle$ D,

and $\angle$ B = $\angle$ E,

$\therefore$ $\angle$ A + $\angle$ B = $\angle$ D + $\angle$ E.

But $\angle$ A + $\angle$ B + $\angle$ C = $\angle$ D + $\angle$ E + $\angle$ F. (I—10, p. 45.)

$\therefore$ $\angle$ C = $\angle$ F.

Case II. Suppose $\angle$ A not = $\angle$ D. (Fig. 2.)

Make $\angle$ **EDG** = $\angle$ **BAC**, and produce its arm to meet **EF**, produced if necessary, at **G**.

$$\text{In } \triangle\text{s ABC, DEG,} \quad \left\{ \begin{array}{l} \angle \mathbf{A} = \angle \mathbf{EDG}, \\ \angle \mathbf{B} = \angle \mathbf{E}, \\ \mathbf{AB} = \mathbf{DE}, \end{array} \right.$$

$$\left. \begin{array}{r} \therefore \angle \mathbf{C} = \angle \mathbf{G}, \\ \text{and } \mathbf{AC} = \mathbf{DG}. \end{array} \right\} \qquad \text{(I—14, p. 54.)}$$

$$\text{But } \mathbf{DF} = \mathbf{AC}, \qquad (\textit{Hyp.})$$

$$\therefore \mathbf{DF} = \mathbf{DG}.$$

$$\therefore \angle \mathbf{DFG} = \angle \mathbf{G}. \qquad \text{(I—3, p. 20.)}$$

$$\text{But } \angle \mathbf{C} = \angle \mathbf{G}.$$

$$\therefore \angle \mathbf{C} = \angle \mathbf{DFG}.$$

$$\angle \mathbf{DFG} + \angle \mathbf{DFE} = \text{two rt. } \angle\text{s},$$

$$\therefore \angle \mathbf{C} + \angle \mathbf{DFE} = \text{two rt. } \angle\text{s}.$$

NOTE.—There are six parts in a triangle, viz., three sides and three angles, and in the cases in which the congruence of two triangles has been established three parts of one triangle, one at least a side, have been given respectively equal to the corresponding parts of the other.

The following general cases occur :—

1. Two sides and the contained angle. The triangles are congruent—Theorem 2.

2. Three sides. The triangles are congruent — Theorem 4.

3. Two angles and a side. The triangles are congruent—Theorem 14.

4. Two sides and an angle opposite one of them. In this case the triangles are congruent if the angle is opposite the greater of the two sides—§ 60, Ex. 3, but

if the angle is opposite the less of the two sides, they are not necessarily congruent—Theorem 15.

5. Three angles. The triangles are not necessarily congruent—§ 60, Ex. 7.

60.—Exercises

1. **If two rt.-∠d △s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the △s are congruent.**

2. If the bisector of the vertical ∠ of a △ also bisects the base, the △ is isosceles.

3. If two △s have two sides of one respectively equal to two sides of the other and the ∠s opposite the greater pair of equal sides equal to each other, the △s are congruent.

4. Construct a △ having given two sides and the ∠ opposite one of them.

   When will there be: (*a*) no solution, (*b*) two solutions, (*c*) only one solution?

5. If two ∠s of a △ be bisected and the bisectors be produced to meet, the line joining the point of intersection to the vertex of the third ∠ bisects that third ∠. Hence.— **The bisectors of the three ∠s of a △ pass through one point.**

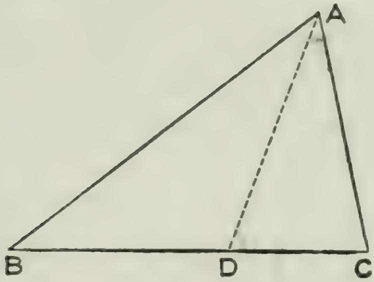
6. If two exterior ∠s of a △ be bisected and the bisectors be produced to meet, the line joining the point of intersection of the bisectors to the vertex of the third ∠ of the △ bisects that third ∠.

7. Draw diagrams to show that if the three ∠s of one △ are respectively equal to the three ∠s of another △, the two △s are not necessarily congruent.

INEQUALITIES

THEOREM 16

Any two sides of a triangle are together greater than the third side.



Hypothesis.—ABC is a $\triangle$.

To prove that $AB + AC > BC$.

Construction.—Bisect $\angle$ A and let the bisector meet BC at D.

Proof.—ADC is an exterior $\angle$ of $\triangle$ ABD,

$\therefore \angle ADC > \angle BAD$. (I—10, Cor., p. 45.)

But $\angle BAD = \angle DAC$.

$\therefore \angle ADC > \angle DAC$.

$\therefore AC > DC$. (I—12, p. 50.)

Similarly it may be shown that

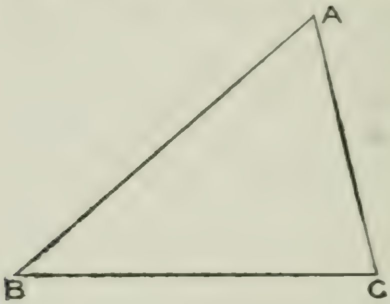
$AB > BD$.

$\therefore AB + AC > BD + DC$,

i.e., $AB + AC > BC$.

In the same manner it may be shown that $AB + BC > AC$ and that $AC + CB > AB$.

Cor.—The difference between any two sides of a triangle is less than the third side.



ABC is a △.

It is required to show that AB - AC < BC.

AB < AC + BC. (I—16, p. 59.)

From each of these unequals take AC,

and AB - AC < BC.

In the same manner it may be shown that AB - BC < AC and that BC - AC < AB.

61.—Exercises

1. Show that the sum of any three sides of a quadrilateral is greater than the fourth side.

2. The sum of the four sides of a quadrilateral is greater than the sum of its diagonals.

3. The sum of the diagonals of a quadrilateral is greater than the sum of either pair of opposite sides.

4. The sum of the st. lines joining any point, except the intersection of the diagonals, to the four vertices of a quadrilateral, is greater than the sum of the diagonals.

5. **If any point within a △ be joined to the ends of a side of the △, the sum of the joining lines is less than the sum of the other two sides of the △.**

6. If any point within a $\triangle$ be joined to the three vertices of the $\triangle$, the sum of the three joining lines is less than the perimeter of the $\triangle$, but greater than half the perimeter.

7. The sum of any two sides of a $\triangle$ is greater than twice the median drawn to the third side.

8. The median of a $\triangle$ divides the vertical $\angle$ into parts, of which the greater is adjacent to the less side.

9. The perimeter of a $\triangle$ is greater than the sum of the three medians.

10. A and B are two fixed points, and CD is a fixed st. line. Find the point P in CD, such that PA + PB is the least possible;

(*a*) When A and B are on opposite sides of CD;

(*b*) When A and B are on the same side of CD.

11. A and B are two fixed points, and CD is a fixed st. line. Find the point P in CD, such that the difference between PA and PB is the least possible;

(*a*) When A and B are on the same side of CD;

(*b*) When A and B are on opposite sides of CD.

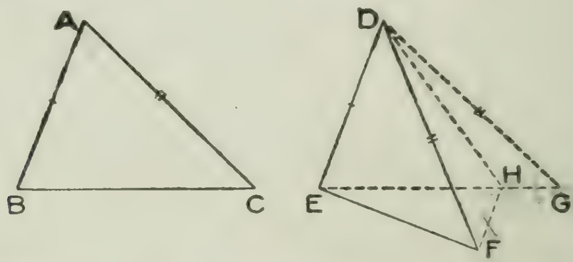
12. Prove Theorem 16 by producing BA to E, making AE = AC, and joining EC.

13. Prove that the shortest line which can be drawn with its ends on the circumferences of two concentric circles, will, when produced, pass through the centre.

14. Prove the Corollary under Theorem 16, (*a*) by cutting off from AB a part AD = AC and joining DC; (*b*) by producing AC to E making AE = AB and joining BE.

Theorem 17

If two triangles have two sides of one respectively equal to two sides of the other but the contained angle in one greater than the contained angle in the other, the base of the triangle which has the greater angle is greater than the base of the other.



Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and $\angle$ BAC $>$ $\angle$ EDF.

To show that BC $>$ EF.

Construction.—Make $\angle$ EDG = $\angle$ BAC and cut off DG = AC, or DF. Join EG. Bisect $\angle$ FDG and let the bisector meet EG at H. Join FH.

Proof.—

In $\triangle$s ABC, DEG, $\left\{\begin{array}{l} \text{AB} = \text{DE}, \\ \text{AC} = \text{DG}, \\ \angle \text{A} = \angle \text{EDG}, \end{array}\right.$

$\therefore$ BC = EG. (I—2, p. 16.)

In $\triangle$s FDH, GDH, $\left\{\begin{array}{l} \text{DF} = \text{DG}, \\ \text{DH is common}, \\ \angle \text{FDH} = \angle \text{GDH}, \end{array}\right.$

$\therefore$ FH = HG.

In $\triangle$ EHF, EH + HF $>$ EF. (I—16, p. 59.)

But $HF = HG$,

$\therefore EH + HG > EF$.

i.e., $EG > EF$.

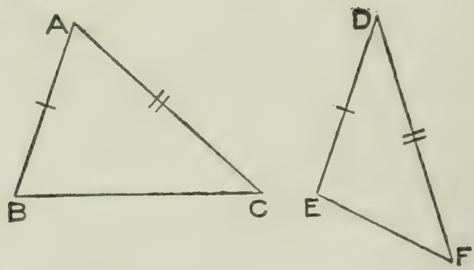
But $BC = EG$,

$\therefore BC > EF$.

Theorem 18

(Converse of Theorem 17)

If two triangles have two sides of one respectively equal to two sides of the other but the base of one greater than the base of the other, the triangle which has the greater base has the greater vertical angle.



Hypothesis.—ABC, DEF are two $\triangle$s having AB = DE, AC = DF and BC > EF.

To prove that $\angle A > \angle D$.

Proof.— If $\angle A$ is not $> \angle D$,

either $\angle A = \angle D$,

or $\angle A < \angle D$.

(1) If $\angle A = \angle D$.

In $\triangle$s ABC, DEF, $\begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$

$\therefore BC = EF$. (I—2, p. 16.)

But this is not so.

$\therefore \angle A$ is not $= \angle D$.

(2) If $\angle A < \angle D$.

$$\text{In } \triangle\text{s ABC, DEF, } \begin{cases} AB = DE, \\ AC = DF, \\ \angle A < \angle D, \end{cases}$$

$$\therefore BC < EF. \qquad \text{(I—17, p. 62.)}$$

But this is not so.

$$\therefore \angle A \text{ is not } < \angle D.$$

Then since $\angle A$ is neither $=$ nor $< \angle D$,

$$\therefore \angle A > \angle D.$$

62.—Exercises

1. ABCD is a quadrilateral having $AB = CD$ and $\angle BAD > \angle ADC$. Show that $\angle BCD > \angle ABC$.

2. In $\triangle$ ABC, $AB > AC$ and D is the middle point of BC. If any point P in the median AD be joined to B and C, $BP > CP$.

If AD be produced to any point Q show that $BQ < QC$.

3. D is a point in the side AB of the $\triangle$ ABC. AC is produced to E making $CE = BD$. BE and CD are joined. Show that $BE > CD$.

4. If two chords of a circle be unequal the greater subtends the greater angle at the centre.

5. Two circles have a common centre at O. A, B are two points on the inner circumference and C, D two on the outer. $\angle AOC > \angle BOD$. Show that $AC > BD$.

6. CD bisects AB at rt. $\angle$s. A point E is taken not in CD. Prove that EA, EB are unequal.

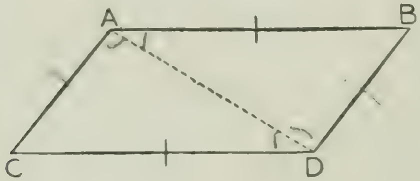
7. In $\triangle$ ABC, $AB > AC$. Equal distances BD, CE are cut off from BA, CA respectively. Prove $BE > CD$.

8. In $\triangle$ ABC, $AB > AC$. AB, AC are produced to D, E making $BD = CE$. Prove $CD > BE$.

Parallelograms

Theorem 19

Straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Hypothesis.—AB, CD are = and ∥.

To prove that (1) AC = BD,

(2) AC ∥ BD.

Construction.—Join AD.

Proof.— ∵ AB ∥ CD,

and AD is a transversal,

∴ ∠ BAD = ∠ CDA. (I—8, p. 40.)

In △s BAD, CDA, $\left\{ \begin{array}{l} \text{BA} = \text{CD}, \\ \text{AD is common}, \\ \angle\ \text{BAD} = \angle\ \text{CDA}, \end{array} \right.$

$\left. \begin{array}{r} \therefore\ \text{BD} = \text{AC}, \\ \text{and } \angle\ \text{BDA} = \angle\ \text{CAD}, \end{array} \right\}$ (I—2, p. 16.)

∵ transversal AD

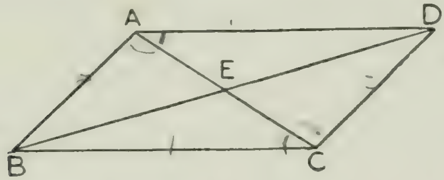
makes ∠ BDA = ∠ CAD,

∴ BD ∥ AC. (I—6, p. 36.)

THEOREM 20

In any parallelogram:

(1) The opposite sides are equal;
(2) The opposite angles are equal;
(3) The diagonal bisects the area;
(4) The diagonals bisect each other.



Hypothesis.—ABCD is a $\parallel$gm, AC, BD its diagonals.

To prove that (1) AD = BC and AB = CD.

(2) $\angle$ BAD = $\angle$ BCD and $\angle$ ABC = $\angle$ ADC.

(3) $\triangle$ ABC = $\triangle$ ACD.

(4) AE = EC and BE = ED.

Proof.—$\because$ AC cuts $\parallel$ lines AD, BC,

$\therefore$ $\angle$ DAC = $\angle$ ACB. (I—8, p. 40.)

$\because$ AC cuts $\parallel$ lines DC, AB,

$\therefore$ $\angle$ DCA = $\angle$ CAB.

In $\triangle$s ACD, ACB, $\begin{cases} \angle \text{DAC} = \angle \text{ACB}, \\ \angle \text{DCA} = \angle \text{CAB}, \\ \text{AC is common,} \end{cases}$

$\therefore$ (1) AD = BC, and CD = AB,
(2) also $\angle$ ADC = $\angle$ ABC,
(3) and $\triangle$ ADC = $\triangle$ ABC. (I—14, p. 54.)

Similarly it may be shown that $\angle$ BAD = $\angle$ BCD.

In $\triangle$s AED, BEC, $\begin{cases} \text{AD} = \text{BC}, \\ \angle \text{DAE} = \angle \text{BCE}, \\ \angle \text{ADE} = \angle \text{CBE}, \end{cases}$

(4) $\therefore$ AE = EC,
and DE = EB. (I—14, p. 54.)

63. **Definitions**.—A parallelogram of which the angles are right angles is called a **rectangle**.

A rectangle of which all the sides are equal to each other is called a **square.**

A figure bounded by more than four straight lines is called a **polygon.**

The name polygon is sometimes used for a figure having any number of sides.

A polygon in which all the sides are equal to each other and all the angles are equal to each other is called a **regular polygon.**

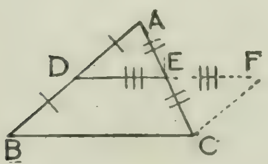
64.—Exercises

1. The diagonals of a rectangle are equal to each other.

2. If the diagonals of a ||gm are equal to each other, the ||gm is a rectangle.

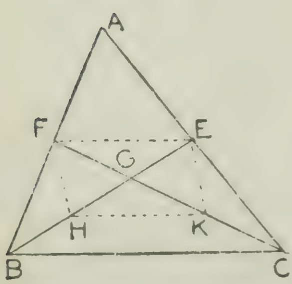
3. A rectangle has two axes of symmetry.

4. A square has four axes of symmetry.



5. **The st. line joining the middle points of the sides of a △ is ‖ the base, and equal to half of it.**

NOTE.—D, E *are the middle points of* AB, AC. *Produce* DE *to* F *making* EF = DE. *Join* FC.



6. **Of two medians of a △ each cuts the other at the point of trisection remote from the vertex.**

NOTE.—*Medians* BE, CF *cut at* G. *Bisect* BG, CG *at* H, K. *Join* FH, HK, KE, EF.

7. **The medians of a △ pass through one point.**

Definition.—The point where the medians of a △ intersect is called the **centroid** of the △.

8. **A st. line drawn through the middle point of one side of a △, ‖ to a second side, bisects the third side.**

9. In any ‖gm the diagonal which joins the vertices of the obtuse ∠s is shorter than the other diagonal.

10. If two sides of a quadrilateral be ‖, and the other two be equal to each other but not ‖, the diagonals of the quadrilateral are equal.

11. Through a given point draw a st. line, such that the part of it intercepted between two given ‖ st. lines is equal to a given st. line.

Show that, in general, two such lines can be drawn.

12. Through a given point draw a st. line that shall be equidistant from two other given points.

Show that, in general, two such lines can be drawn.

13. Draw a st. line ‖ to a given st. line, and such that the part of it intercepted between two given intersecting lines is equal to a given st. line.

14. **BAC** is a given ∠, and **P** is a given point. Draw a st. line terminated in the st. lines **AB**, **AC** and bisected at **P**.

15. Construct a △ having given the middle points of the three sides.

16. If the diagonals of a ‖gm cut each other at rt. ∠s, the ‖gm is a rhombus.

17. Every st. line drawn through the intersection of the diagonals of a ‖gm, and terminated by a pair of opposite sides, is bisected, and bisects the ‖gm.

18. Bisect a given ‖gm by a st. line drawn through a given point.

19. Divide a given △ into four congruent △s.

20. The bisectors of two opposite $\angle$s of a $\|$gm are $\|$ to each other.

21. In the quadrilateral **ABCD**, **AB** $\|$ **CD** and **AD** = **BC**. Prove that (1) $\angle$ **C** = $\angle$ **D**; (2) if **E**, **F** are the middle points of **AB**, **CD** respectively, **EF** $\perp$ **AB**.

22. **On a given st. line construct a square.**

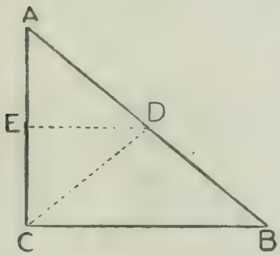
23. Construct a square having its diagonal equal to a given st. line.

24. **ABC** is a $\triangle$ and **DE** a st. line. Draw a st. line = **DE**, $\|$ **BC** and terminated in **AB**, **AC**, or in these lines produced.

25. Inscribe a rhombus in a given $\|$gm, such that one vertex of the rhombus is at a given point in a side of the $\|$gm.

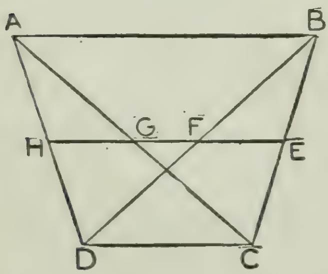
26. **ABC** is an isosceles $\triangle$ in which **AB** = **AC**. From **P**, any point in **BC**, **PX**, **PY** are drawn $\perp$ **AB**, **AC** respectively and **BM** is $\perp$ **AC**. Prove that **PX** + **PY** = **BM**.

If **P** is taken on **CB** produced, prove that **PY** − **PX** = **BM**.



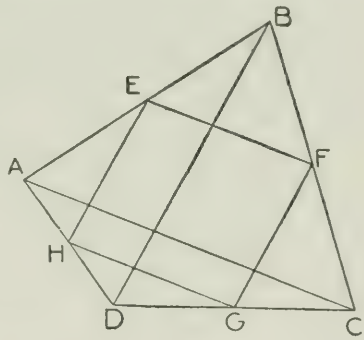
27. The middle point of the hypotenuse of a rt.-$\angle$d $\triangle$ is equidistant from the three vertices.

NOTE.—*Through* **D**, *the middle point of the hypotenuse* **AB**, *draw* **DE** $\|$ **BC**. *Join* **DC**.



28. **ABCD** is a quadrilateral in which **AB** $\|$ **CD**. **E**, **F**, **G**, **H** are the middle points of **BC**, **BD**, **AC**, **AD**. Prove that: (1) the st. line through **E** $\|$ **AB**, or **DC**, passes through **F**, **G** and **H**; (2) **HE** = half the sum of **AB** and **DC**; (3) **GF** = half the difference of **AB** and **DC**.

29. **E, F, G, H** are the middle points of the sides **AB, BC, CD, DA** of the quadrilateral **ABCD.** Prove that **EFGH** is a ||gm. Show also that: (1) the perimeter of **EFGH** = **AC** + **BD**; (2) if **AC** = **BD, EFGH** is a rhombus; (3) if **AC** ⊥ **BD, EFGH** is a rectangle; (4) if **AC** = and ⊥ **BD, EFGH** is a square.

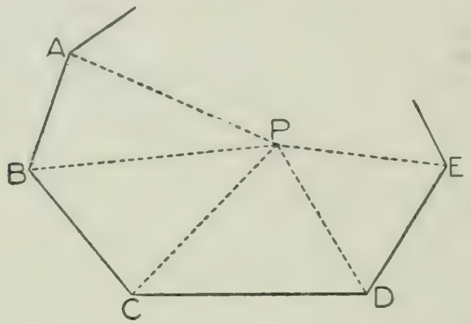


30. The middle points of a pair of opposite sides of a quadrilateral and the middle points of the diagonals are the vertices of a ||gm.

31. The st. lines joining the middle points of the opposite sides of a quadrilateral and the st. line joining the middle points of the diagonals are concurrent.

Theorem 21

The sum of the interior angles of a polygon of n sides is $(2n-4)$ right angles.



Hypothesis.—**ABCDE**, etc., is a closed polygon of n sides.

To prove that the sum of the interior angles is $(2n - 4)$ rt. $\angle$s.

Construction.—Take any point **P** within the polygon and join **P** to the vertices.

Proof.—The polygon is divided into n $\triangle$s **PAB, PBC, PCD,** etc.

The sum of the interior $\angle$s of each $\triangle$ is two rt. $\angle$s. (I—10, p. 45.)

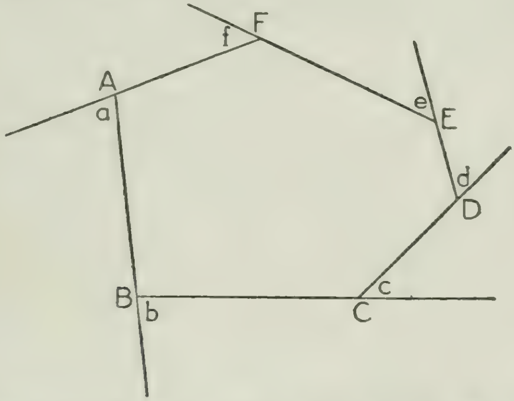
$\therefore$ the sum of the $\angle$s of the n $\triangle$s is $2n$ rt. $\angle$s.

But the $\angle$s of the n $\triangle$s make up the interior $\angle$s of the polygon together with the $\angle$s about the point **P.**

And the sum of the $\angle$s about **P** equals 4 rt. $\angle$s.

$\therefore$ the sum of the interior $\angle$s of the polygon $=$ $(2n - 4)$ rt. $\angle$s.

Cor.—If the sides of a polygon are produced in order, the sum of the exterior angles thus formed is four right angles.



If the polygon has n sides, the sum of all the st. $\angle$s at the vertices = $2n$ rt. $\angle$s.

But, the sum of the interior $\angle$s = $(2n - 4)$ rt. $\angle$s. (I—21, p. 72.)

$\therefore$, subtracting, $\angle a + \angle b +$ etc. = 4 rt. $\angle$s.

65.—Exercises

1. Find the number of degrees in an exterior $\angle$ of an equiangular polygon of twelve sides.

Hence, find the number of degrees in each interior $\angle$.

2. Find the number of degrees in each $\angle$ of (*a*) an equiangular pentagon; (*b*) an equiangular hexagon; (*c*) an equiangular octagon; (*d*) an equiangular decagon.

3. Each $\angle$ of an equiangular polygon contains 162°. Find the number of sides.

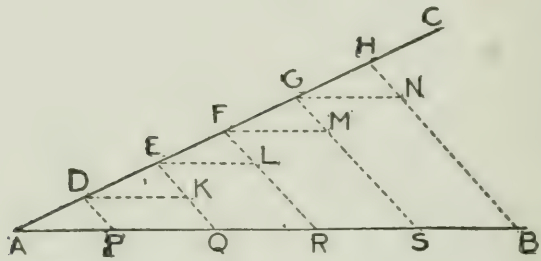
4. Each $\angle$ of an equiangular polygon contains 170°. Find the number of sides.

5. Show that the space around a point may be exactly filled in by six equilateral $\triangle$s, four squares, or three equiangular hexagons. Draw the diagram in each case.

Construction

Problem 8

To divide a straight line into any number of equal parts.



Let AB be the given st. line.

To divide AB into five equal parts.

Construction.—From A draw a st. line AC.

From AC cut off five equal parts AD, DE, EF, FG, GH.

Join HB.

Through D, E, F, G draw lines ∥ HB cutting AB at P, Q, R, S.

AB is divided into five equal parts at P, Q, R, S.

Proof.—Through D, E, F, G draw DK, EL, FM, GN ∥ AB.

∵ AE cuts the parallels AP, DK,

∴ ∠ EDK = ∠ DAP. (I—9, p. 42.)

∵ AE cuts the parallels DP, EQ,

∴ ∠ ADP = ∠ DEQ.

In △s ADP, DEK, $\left\{\begin{array}{l} \angle \text{ DAP} = \angle \text{ EDK}, \\ \angle \text{ ADP} = \angle \text{ DEK}, \\ \text{AD} = \text{DE}, \end{array}\right.$

∴ AP = DK (I—14, p. 54.)

But PQ = DK. (I—20, p. 67.)

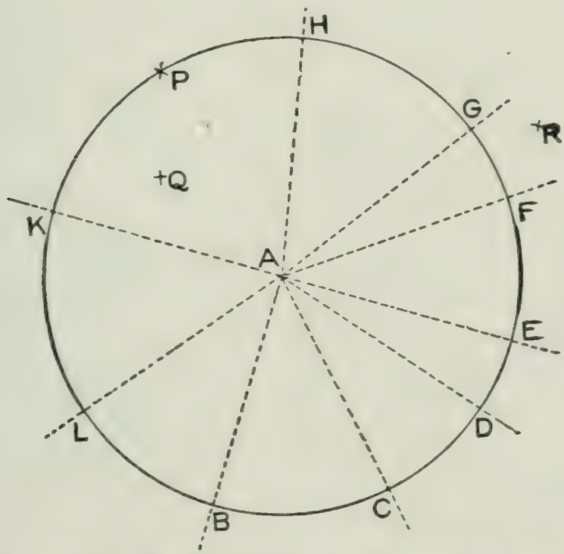
∴ PQ = AP.

Similarly it may be shown that each of QR, RS, SB = AP.

By this method a st. line may be divided into any number of equal parts.

Loci

66. **Example I.**—A is a point and from A straight lines are drawn in different directions in the same plane.



On each line a distance of one inch is measured from A and the resulting points are B, C, D, etc.

Is there any one line that contains all of the points in the plane that are at a distance of one inch from A?

To answer this question describe a circle with centre A and radius one inch. The circumference of this circle is a line that passes through all the points.

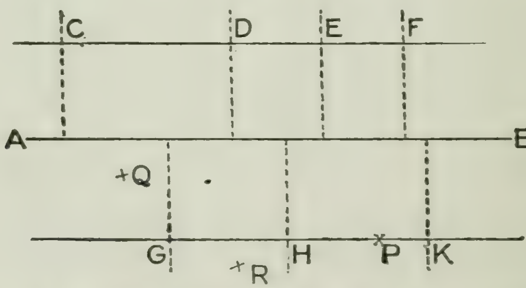
Mark any other point P on the circumference. What is the distance of P from A? From the definition of a circle the answer to this question is one inch.

If any point Q be taken within the circle, its distance from A is less than one inch, and if any point R be taken without the circle, its distance from A is greater than one inch.

Thus every point in the circumference satisfies the condition of being just one inch from A, and no point, in the plane, that is not on the circumference does satisfy this condition.

This circumference is called the **locus** of all points in the plane that are at a distance of one inch from A.

Example 2:—AB is a straight line of indefinite length, to which any number of perpendiculars are drawn.



On each of these perpendiculars a distance of one centimetre is measured from AB, and the resulting points are C, D, E, etc.

Are there any lines that contain all of the points, such as C, D, etc., that are at a distance of one centimetre from AB?

Draw two straight lines parallel to AB, each at a distance of one centimetre from AB, and one or other of these lines will pass through each of the points.

Any point P in CF, or in GK, is at a distance of one centimetre from AB; any point Q in the space between CF and GK is less than one centimetre from AB, and any point R in the plane and neither between CF and GK nor in one of these lines is more than one centimetre from AB.

Thus every point in CF and GK satisfies the condition of being just one centimetre from AB, and no point outside of these lines and in the plane does satisfy this condition.

The two lines GF, GK make up the **locus** of all points in the plane that are at a distance of one centimetre from AB.

Definition.—When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

67. In place of speaking of the "locus of the points which satisfy a given condition," the alternative expression "locus of the point which satisfies a given condition" may be used.

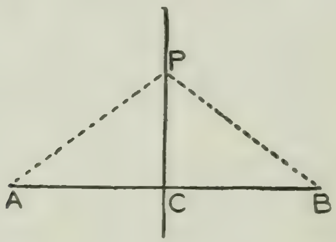
Suppose a point to move in a plane so that it traces out a continuous line, but its distance from a fixed point A in the plane is always one inch; then it must move on the circumference of the circle of centre A and radius one inch, and the locus of the point in its different positions is that circumference.

The following definition of a locus may thus be given as an alternative to that in § 66.

Definition.—**If a point moves on a line, or on lines, so that it constantly satisfies a given condition, the figure consisting of the line, or lines, is the locus of the point.**

Theorem 22

The locus of a point which is equidistant from two given points is the right bisector of the straight line joining the two given points.



Hypothesis.—P is a point equidistant from A and B.

To prove that P is on the right bisector of AB.

Construction.—Bisect AB at C.

Join PC, PA, PB.

Proof.—

In $\triangle$s PAC, PBC, $\left\{\begin{array}{l} \text{PA} = \text{PB}, \\ \text{AC} = \text{CB}, \\ \text{PC is common,} \end{array}\right.$

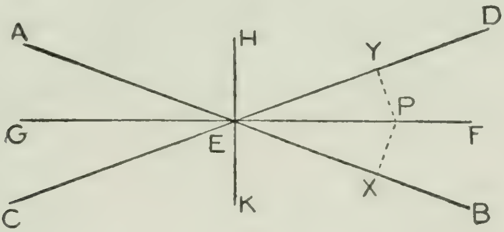
$\therefore \triangle \text{PAC} \equiv \triangle \text{PCB},$ (I—4, p. 22.)

$\therefore \angle \text{PCA} = \angle \text{PCB},$

and $\therefore$ P is on the right bisector of AB.

THEOREM 23

The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the two straight lines.



Hypothesis. — AB, CD are two st. lines cutting at E; GF, HK are the bisectors of $\angle$s made by AB, CD.

To prove that the locus of a point equidistant from AB, and CD consists of GF and HK.

Construction. — Take any point P in GF. Draw PX $\perp$ AB, PY $\perp$ CD.

Proof.—

In $\triangle$s PEX, PEY, $\left\{\begin{array}{l}\angle \text{PEX} = \angle \text{PEY}, \\ \angle \text{PXE} = \angle \text{PYE}, \\ \text{PE is common,}\end{array}\right.$

$\therefore$ PX = PY. (I—14, p. 54.)

$\therefore$ every point in GF is equidistant from AB and CD.

Similarly it may be shown that every point in HK is equidistant from AB and CD.

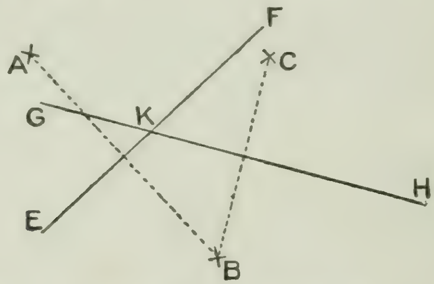
$\therefore$ the locus of points equidistant from AB, CD consists of GF and HK.

68. **Problem:—To find the point that is equally distant from three given points, that are not in the same straight line.**

Let A, B, C be the three given points.

It is required to find a point equally distant from A, B and C.

Draw EF the locus of all points that are equally distant from A and B. (I—22, p. 78.)



Draw GH the locus of all points that are equally distant from B and C.

Let EF and GH meet at K.

Then K is the required point.

K is on EF, ∴ KA = KB.

K is on GH, ∴ KB = KC.

Consequently K is equally distant from A, B and C.

69.—Exercises

1. Find the locus of the centres of all circles that pass through two given points.

2. Describe a circle to pass through two given points and have its centre in a given st. line.

3. Describe a circle to pass through two given points and have its radius equal to a given st. line. Show that

generally two such circles may be described. When will there be only one? and when none?

4. Find the locus of a point which is equidistant from two given || st. lines.

5. In a given st. line find two points each of which is equally distant from two given intersecting st. lines.

When will there be only one solution?

6. Find the locus of the vertices of all △s on a given base which have the medians drawn to the base equal to a given st. line.

7. Find the locus of the vertices of all △s on a given base which have one side equal to a given st. line.

8. Construct a △ having given the base, the median drawn to the base, and the length of one side.

9. Find the locus of the vertices of all △s on a given base which have a given altitude.

10. Construct a △ having given the base, the median drawn to the base, and the altitude.

11. Construct a △ having given the base, the altitude and one side.

12. Find the locus of a point such that the sum of its distances from two given intersecting st. lines is equal to a given st. line.

13. Find the locus of a point such that the difference of its distances from two given intersecting st. lines is equal to a given st. line.

14. Find the locus of the vertices of all △s on a given base which have the median drawn from one end of the base equal to a given st. line.

15. Show that, if the ends of a st. line of constant length slide along two st. lines at rt. ∠s to each other, the locus of its middle point is a circle.

16. **AB** is a st. line and **C** is a point at a distance of 2 cm. from **AB**. Find a point which is 1 cm. from **AB** and 4 cm. from **C**. How many such points can be found?

17. Two st. lines, **AB, CD,** intersect each other at an ∠ of 45°. Find all the points that are 3 cm. from **AB** and 2 cm. from **CD**.

18. **ABC** is a scalene △. Find a point equidistant from **AB** and **AC**, and also equidistant from **B** and **C**.

19. Find a point equidistant from the three vertices of a given △.

20. Find four points each of which is equidistant from the three sides of a △.

Note.—*Produce each side in both directions.*

21. Find the locus of a point at which two equal segments of a st. line subtend equal ∠s.

22. Find the locus of the centre of a circle which shall pass through a given point and have its radius equal to a given st. line.

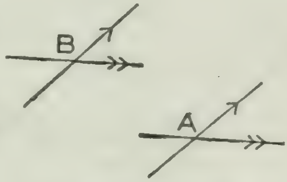
23. A st. line of constant length remains always ∥ to itself, while one of its extremities describes the circumference of a fixed circle. Find the locus of the other extremity.

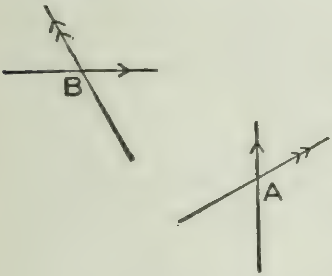
24. The locus of the middle points of all st. lines drawn from a fixed point to the circumference of a fixed circle is a circle.

Miscellaneous Exercises

1. If a st. line be terminated by two $\parallel$s, all st. lines drawn through its middle point and terminated by the same $\parallel$s are bisected at that point.

2. **If two lines intersecting at A be respectively $\parallel$ to two lines intersecting at B, each $\angle$ at A is either equal to or supplementary to each $\angle$ at B.**





3. **If two lines intersecting at A be respectively $\perp$ to two lines intersecting at B, each $\angle$ at A is either equal to or supplementary to each $\angle$ at B.**

4. If from any point in the bisector of an $\angle$ st. lines be drawn $\parallel$ to the arms of the $\angle$ and terminated by the arms, these st. lines are equal to each other.

5. In the base of a $\triangle$ find a point such that the st. lines drawn from that point $\parallel$ to the sides of the $\triangle$ and terminated by the sides are equal to each other.

6. One $\angle$ of an isosceles $\triangle$ is half each of the others. Calculate the $\angle$s.

7. If the $\perp$ from the vertex of a $\triangle$ to the base falls within the $\triangle$, the segment of the base adjacent to the greater side of the $\triangle$ is the greater.

8. If a star-shaped figure be formed by producing the alternate sides of a polygon of n sides, the sum of the $\angle$s at the points of the star is $(2n - 8)$ rt. $\angle$s.

9. In a quadrilateral ABCD, $\angle$ A = $\angle$ B and $\angle$ C = $\angle$ D. Prove that AD = BC.

10. The bisectors of the $\angle$s of a ||gm form a rectangle, the diagonals of which are || to the sides of the original ||gm; and equal to the difference between them.

11. From **A**, **B** the ends of a st. line $\perp$s **AC**, **BD** are drawn to any st. line. **E** is the middle point of **AB**. Show that **EC** = **ED**.

12. If through a point within a $\triangle$ three st. lines be drawn from the vertices to the opposite sides, the sum of these st. lines is greater than half the perimeter of the $\triangle$.

13. **A**, **D** are the centres of two circles, and **AB**, **DE** are two || radii. **EB** cuts the circumferences again at **C**, **F**. Show that **AC** || **DF**.

14. The bisectors of the interior $\angle$s of a quadrilateral form a quadrilateral of which the opposite $\angle$s are supplementary.

15. In a given square inscribe an equilateral $\triangle$ having one vertex at a vertex of the square.

16. Through two given points draw two st. lines, forming an equilateral $\triangle$ with a given st. line.

17. Draw an isosceles $\triangle$ having its base in a given st. line, its altitude equal to a given st. line, and its equal sides passing through two given points.

18. If a $\perp$ be drawn from one end of the base of an isosceles $\triangle$ to the opposite side, the $\angle$ between the $\perp$ and the base = half the vertical $\angle$ of the $\triangle$.

19. If any point **P** in **AD** the bisector of the $\angle$ **A** of $\triangle$ **ABC** be joined to **B** and **C**, the difference between **PB** and **PC** is less than the difference between **AB** and **AC**.

20. If any point **P** in the bisector of the exterior $\angle$ at **A** in the $\triangle$ **ABC** be joined to **B** and **C**, **PB** + **PC** > **AB** + **AC**.

21. **BAC** is a rt. ∠ and **D** is any point. **DE** is drawn ⊥ **AB** and produced to **F**, making **EF** = **DE**. **DG** is drawn ⊥ **AC** and produced to **H**, making **GH** = **DG**. Show that **F, A, H** are in the same st. line.

22. Construct a △ having its perimeter equal to a given st. line and its ∠s respectively equal to the ∠s of a given △.

23. In any quadrilateral, the sum of the exterior ∠s at one pair of opposite vertices = the sum of the interior ∠s at the other vertices.

24. If the arms of one ∠ be respectively ∥ to the arms of another ∠, the bisectors of the ∠s are either ∥ or ⊥.

25. In a given △ inscribe a ∥gm the diagonals of which intersect at a given point.

26. Show that the ⊥s from the centre of a circle to two equal chords are equal to each other.

27. Construct a quadrilateral having its sides equal to four given st. lines and one ∠ equal to a given ∠.

28. The bisector of ∠ **A** of △ **ABC** meets **BC** at **D** and **BC** is produced to **E**. Show that ∠ **ABC** + ∠ **ACE** = twice ∠ **ADC**.

29. The bisectors of ∠s **A** and **B** of △ **ABC** intersect at **D**. Show that ∠ **ADB** = 90° + half of ∠ **C**.

30. The sides **AB, AC** of a △ **ABC** are bisected at **D, E**; and **BE, CD** are produced to **F, G**, so that **EF** = **BE** and **DG** = **CD**. Show that **F, A, G** are in the same st. line, and that **FA** = **AG**.

31. **ABC** is an isosceles △, having **AB** = **AC**. **AE, AD** are equal parts cut off from **AB, AC** respectively. **BD, CE** cut at **F**. Show that **FBC** and **FDE** are isosceles △s.

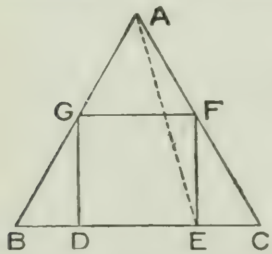
32. In a $\triangle$ **ABC**, the bisector of $\angle$ **A** and the right bisector of **BC** meet at **D**. **DE**, **DF** are drawn $\perp$ **AB**, **AC** respectively. Show that the point **D** is not within the $\triangle$, that **AE** = **AF** and that **BE** = **CF**.

33. **ABCD** is a quadrilateral having $\angle$ **B** = $\angle$ **C** and **AB** $<$ **CD**. Prove that $\angle$ **A** $>$ $\angle$ **D**.

34. Through a given point draw a st. line cutting two intersecting st. lines and forming an isosceles $\triangle$ with them.

Show that two such lines can be drawn through the given point.

35. If **ACB** be a st. line and **ACD**, **BCD** two adjacent $\angle$s, any $\parallel$ to **AB** will meet the bisectors of these $\angle$s in points equally distant from where it meets **CD**.



36. Inscribe a square in a given equilateral $\triangle$.

NOTE.—*Draw a sketch as in the diagram given here. Join* **AE**.

What is the number of degrees in $\angle$ **CAE**?

37. **ABC** is a $\triangle$, **AX** is $\perp$ **BC**, and **AD** bisects $\angle$ **BAC**. Show that $\angle$ **XAD** equals half the difference of $\angle$s **B** and **C**.

38. Construct a $\parallel$gm having its diagonals and a side respectively equal to three given st. lines.

39. Find a point in each of two $\parallel$ st. lines such that the two points are equally distant from a given point and the st. line joining them subtends a rt. $\angle$ at the given point.

40. **P**, **Q** are two given points on the same side of a given st. line **BC**. Find the position of a point **A** in **BC** such that $\angle$ **PAB** = $\angle$ **QAC**.

NOTE.—*If* **P**, **Q** *are two points on a billiard table and* **BC** *the side of the table, a ball starting from* **P** *and reflected from* **BC** at **A** *would pass through* **Q**.

41. Find the path of a billiard ball which, starting from a given point, is reflected from the four sides of the table and passes through another given point.

42. **BAC** is a given ∠ and **D, E** are two given st. lines. Find a point **P** such that its distances from **AB, AC** equal **D, E** respectively.

43. Find in a side of a △ a point such that the sum of the two st. lines drawn from the point ‖ to the other sides and terminated by them is equal to a given st. line.

44. **AEB, CED** are two st. lines, and each of the quadrilaterals **CEAF, BEDG** is a rhombus. Prove that **FEG** is a st. line.

45. **F** is a point within the △ **ABC** such that ∠ **FBC** = ∠ **FCB**. **BF, CF** produced meet **AC, AB** at **D, E** respectively. Prove that if ∠ **AFD** = ∠ **AFE**, △ **ABC** is isosceles.

46. **D** is a point in the base **BC** of an equilateral △ **ABC**. **E** is the middle point of **AD**. Prove that **EC** > **ED**.

47. **ABC** is a △ of which ∠ **BAC** is obtuse, **O** a point within it; **BO, CO** meet **AC, AB** at **D, E** respectively. Prove that **BD** + **CE** > **BE** + **ED** + **DC**.

48. **D, E, F** are points in the sides **BC, CA, AB** of an equilateral △ and are such that **BD** = **CE** = **AF**. If **AD, BE, CF** do not all pass through one point, they form an equilateral △.

49. The bisector of ∠ **A** of △ **ABC** meets **BC** at **D**. **DE, DF** drawn ‖ **AB, AC** respectively meet **AC, AB** at **E, F**. Prove that **AEDF** is a rhombus.

50. Through each angular point of a △ a st. line is drawn ‖ the opposite side: prove that the △ formed by these three st. lines is equiangular to the given △.

51. **AD, BE, CF** respectively bisect the interior ∠ **A** and the exterior ∠s at **B** and **C** of the △ **ABC**. Show that no two of the lines **AD, BE, CF** can be ∥.

52. **DE** is ∥ to the base **AB** of the isosceles △ **CAB** and cuts **CA, CB**, or those sides produced, at **D, E** respectively. **AE, BD** cut at **F**. Prove that **DEF** is an isosceles △.

53. Through **A, B** the extremities of a diameter of a circle ∥ chords **AC, BD** are drawn. Prove that **AC** = **BD**; and that **CD** is a diameter of the circle.

54. The median drawn from the vertex of a △ is >, = or < half the base according as the vertical ∠ is acute, right or obtuse.

55. **ABC** is a △, obtuse-∠d at **C**; st. lines are drawn bisecting **CA, CB** at rt. ∠s, cutting **AB** in **D, E** respectively. Prove that ∠ **DCE** is equal to twice the excess of ∠ **ACB** over a rt. ∠.

56. With one extremity **C** of the base **BC** of an isosceles △ **ABC** as centre, and radius **CB**, a circle is described cutting **AB, AC** at **D, E** respectively. Prove that **DE** ∥ to the bisector of ∠ **B**.

57. In △ **ABC** side **BC** is produced to **D**. Prove that the ∠ between the bisectors of ∠s **ABC, ACD** = half the ∠ **A**.

58. Through the vertices of △ **ABC**, st. lines falling within the △ are drawn making equal ∠s **BAL, CBM, ACN**; if these lines intersect in **D, E, F**, prove △ **DEF** equiangular to △ **ABC**.

59. If the ∠ between two adjacent sides of a ∥gm be increased, while their lengths do not alter, the diagonal through the point of intersection will decrease.

60. **A, B, C** are three given points. Find a point equidistant from **A, B** and such that its distance from **C** equals a given st. line. When is the problem impossible?

61. Through a fixed point draw a st. line which shall make with a given st. line adjacent ∠s the difference of which = a given ∠.

62. Construct a △ having given one ∠ and the lengths of the ⊥s from the vertices of the other ∠s on the opposite sides.

63. Construct an isosceles △ having given the vertical ∠ and the altitude.

64. Construct an isosceles △ having given the perimeter and altitude.

65. Prove that the quadrilateral formed by joining the extremities of two diameters of a circle is a rectangle.

66. In a given ||gm inscribe a rhombus, such that one diagonal passes through a given point.

67. St. lines are drawn from a given point to a given st. line. Find the locus of the middle points of the st. lines.

68. St. lines are drawn from a given point to the circumference of a given circle. Find the locus of the middle points of the st. lines.

69. The sum of the ⊥s from any point within an equilateral △ to the three sides is equal to the altitude of the △.

70. Draw a square which has the sum of a side and a diagonal equal to 3 inches.

71. Draw a square in which the difference between a diagonal and a side is 1 inch.

72. Draw a rectangle having one side 2 inches in length, and subtending an $\angle$ of $40°$ at the point of intersection of the diagonals.

(*Use a protractor in Exercises 72 to 86.*)

73. Draw a ||gm with diagonals 2 inches and 4 inches and their $\angle$ of intersection $50°$.

74. Draw a ||gm with diagonals 4 inches and 7 inches and one side 5 inches.

75. Draw a ||gm with side 3 inches, diagonal $2\frac{1}{2}$ inches and $\angle$ $35°$. Show that there are two solutions.

76. Draw a ||gm with side $2\frac{3}{8}$ inches, $\angle$ $70°$ and diagonal opposite $\angle$ of $70°$ equal to 4 inches.

77. Draw a rectangle having the perimeter 8 inches and an $\angle$ between the diagonals $80°$.

78. Draw a rectangle having the difference of two sides 1 inch and an $\angle$ between the diagonals $50°$.

79. Draw a rectangle which has the perimeter 9 inches and a diagonal $3\frac{1}{2}$ inches.

80. Draw an $\angle$ of $55°$. Find within the $\angle$ a point which is 1 inch from one arm and 2 inches from the other.

81. Construct a $\triangle$ in which side $a = 7$ cm., $b + c =$ 10·6 cm. and $\angle \mathsf{A} = 78°$.

82. Construct a $\triangle$ with perimeter 4 inches and $\angle$s $70°$ and $50°$.

83. **AB**, **CD** are two || st. lines; **P**, **Q** two fixed points. Find a point equidistant from **AB**, **CD** and also equidistant from **P** and **Q**. When is this impossible?

84. Through two given points on the same side of a given st. line draw two st. lines so as to form with the given st. line an equilateral △.

85. Construct a rhombus with one diagonal 2 inches and the opposite ∠ 100°.

86. Construct a △ in which $a = 8$ cm., $b - c = 2$ cm., ∠ **C** = 50°.

87. Squares **ABGE, ACHF** are described externally on two sides of a △ **ABC**. Prove that the median **AD** of the △ is ⊥ **EF** and equal to half of **EF**.

NOTE.—*Rotate* △ **ABC** *through a rt.* ∠ *making* **AC** *coincide with* **AF.**

88. Prove also in Ex. 87 that **EC** is ⊥ and = **BF.**

89. Trisect a rt. ∠.

90. From any point in the base of an isosceles △ st. lines are drawn || to the equal sides and terminated by them. Prove that the sum of these lines = one of the equal sides.

91. **ABC** is a st. line such that **AB** = **BC.** ⊥s are drawn from **A, B, C** to another st. line **EF.** Prove that the ⊥ from **B** = half the sum of the ⊥s from **A** and **C,** unless **EF** passes between **A** and **C,** and then the ⊥ from **B** = half the difference of the ⊥s from **A** and **C.**

92. **AD** is the bisector of ∠ **A** of △ **ABC,** and **M** the middle point of **BC. BE** and **CF** are ⊥ **AD.** Prove that **ME** = **MF.**

93. **E, F** are the middle points of **AD, BC** respectively in the ||gm **ABCD.** Prove that **BE, DF** trisect **AC.**

94. Find a point **P** in the side **AC** of a △ **ABC** so that **AP** may be equal to the ⊥ from **P** to **BC.**

95. If the st. line **AB** be bisected at **C** and produced to **D**, prove that **CD** is half the sum of **AD**, **BD**.

96. In △ **ABC** side **AC** > side **AB**; **AX** ⊥ **BC** and **AD** is a median. Prove that (1) ∠ **CAX** > ∠ **BAX**; (2) ∠ **CAD** < ∠ **DAB**; (3) the bisector of ∠ **BAC** falls between **AX** and **AD**.

97. The median of a △ **ABC** drawn from **A** is not less than the bisector of ∠ **A**.

98. In a quadrilateral **ABCD**, **AB** = **DC** and ∠ **B** = ∠ **C**. Prove that **AD** ∥ **BC**.

99. If two medians of a △ are equal, the △ is isosceles.

NOTE.—*Use Ex. 6, § 64.*

100. If both pairs of opposite ∠s of a quadrilateral are equal, the quadrilateral is a ∥gm.

101. Find the point on the base of a △ such that the difference of the ⊥s from it to the sides is equal to a given st. line.

102. Find the point on the base of a △ such that the sum of the ⊥s from it to the sides is equal to a given st. line.

103. Show that the three exterior ∠s at **A**, **C**, **E**, in the hexagon **ABCDEF**, are together less than the three interior ∠s at **B**, **D**, **F** by two rt. ∠s.

BOOK II

Areas of Parallelograms and Triangles

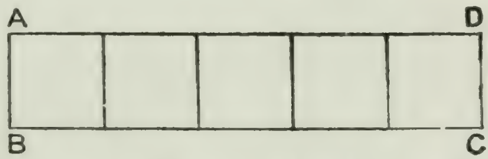
70. A square unit of area is a square, each side of which is equal to a unit of length.

Examples:—A square inch is a square each side of which is one inch; a square centimetre is a square each side of which is one centimetre.

The acre is an exceptional case.

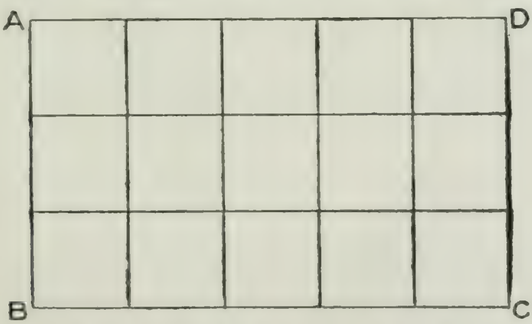
71. A numerical measure of any area is the number of times the area contains some unit of area.

ABCD is a rectangle one centimetre wide and five centimetres long.



This rectangle is a strip divided into five square centimetres, and consequently the numerical measure of its area in square centimetres is 5.

72. **ABCD** is a rectangle 3 cm. wide and 5 cm. long.



This rectangle is divided into 5 strips of 3 sq. cm. each, or into 3 strips of 5 sq. cm. each, and consequently

the measure of the area in square centimetres is 5×3 sq. cm., or 3×5 sq. cm.

Similarly, if the length of a rectangle is 2·34 inches and its breadth ·56 of an inch, the one-hundreth of an inch may be taken as the unit and the rectangle can be divided into 234 strips each containing 56 square one-hundreths of an inch. The measure of the area then is 234×56 of these small squares, ten thousand (100×100) of which make one square inch.

This method of expressing the area of a rectangle may be carried to any degree of approximation, so that in all cases the numerical measure of its area is equal to the product of its length by its breadth.

In a rectangle any side may be called the base, and then either of the adjacent sides is the altitude.

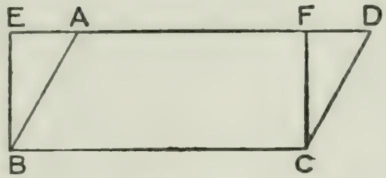
A rectangle, as ABCD, is commonly represented by the symbol AB. BC, where AB and BC may be taken to represent the number of units in the length and the breadth respectively.

Or, if a be the measure of the base of a rectangle and b the measure of its altitude, the area is ab.

In the case of a square, the base is equal to the altitude, and if the measure of each be a, the area is a^2.

THEOREM 1

The area of a parallelogram is equal to that of a rectangle on the same base and of the same altitude.



Hypothesis.—ABCD is a ||gm and EBCF a rectangle on the same base BC and of the same altitude EB.

To prove that the area of the ||gm ABCD = the area of rect. EBCF.

Proof.— $\because$ ED cuts the ||s AB, DC,

$\therefore \angle$ EAB $= \angle$ FDC. (I—9, p. 42.)

$\because$ ABCD is a ||gm,

$\therefore$ AB = CD. (I—20, p. 67.)

In $\triangle$s EAB, FDC, $\begin{cases} \angle \text{EAB} = \angle \text{FDC}, \\ \angle \text{AEB} = \angle \text{DFC}, \\ \text{AB} = \text{DC}, \end{cases}$

$\therefore \triangle$ AEB $= \triangle$ FDC, (I—14, p. 54.)

Figure EBCD $- \triangle$ EAB = ||gm ABCD,

Figure EBCD $- \triangle$ FDC = rect. EBCF;

and as equal parts have been taken from the same area, the remainders are equal.

$\therefore$ ||gm ABCD = rect. EBCF.

Cor.—If *a* be the measure of the base of a ||gm and *b* the measure of its altitude, the area, being the same as that of a rect. of the same base and altitude, = *ab*.

73.—Practical Exercises

1. Draw a ||gm having two adjacent sides 6·4 cm. and 7·3 cm. and the contained ∠ 30°. Find its area.

2. Draw a ||gm having the two diagonals 4·8 cm. and 6·8 cm. and an ∠ between the diagonals 75°. Find its area.

3. The area of a ||gm is 50 sq. cm., one side is 10 cm. and one ∠ is 60°. Construct the ||gm, and measure the other side.

4. Draw a rectangle of base 7 cm. and height 4 cm. On the same base construct a ||gm having the same area as the rectangle and two of its sides each 65 mm. Measure one of the smaller ∠ s of the ||gm.

5. Make a ||gm having sides 10 and 7 cm. and one ∠ 60°. Make a rhombus equal in area to the ||gm and having each side 10 cm. Measure the shorter diagonal of the rhombus.

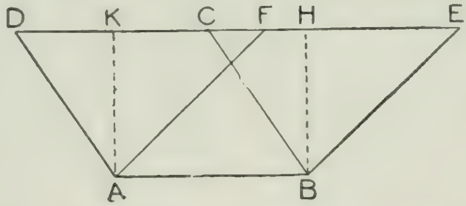
6. Make a rectangle 8 cm. by 5 cm. Construct a ||gm equal in area to the rectangle and having two sides 7 cm. and 8 cm. Construct a rhombus equal in area to the ||gm and having each side 7 cm. Measure the shorter diagonal of the rhombus.

7. Make a rhombus having each side 8 cm. and its area 50 sq. cm. Measure the shorter diagonal.

Answers :—1. 23·4 sq. cm. nearly. 2. 15·8 sq. cm. nearly. 3. 57·7 mm. nearly. 4. 38° nearly. 5. 64 mm. nearly. 6. 64 mm. nearly. 7. 69 mm. nearly.

THEOREM 2

Parallelograms on the same base and between the same parallels are equal in area.



Hypothesis.—ABCD, ABEF are ||gms on the same base AB and between the same ||s AB, DE.

To prove that ||gm ABCD = ||gm ABEF.

Construction.—Draw AK, BH each ⊥ to both AB and DE.

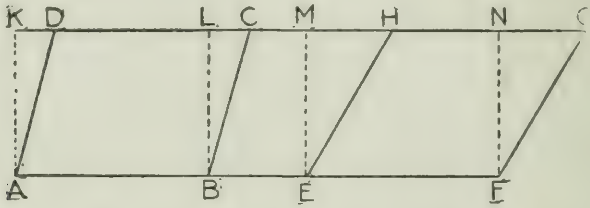
Proof.—∵ ||gm ABCD = rect. ABHK, (II—1, p. 95.)

and ||gm ABEF = rect. ABHK,

∴ ||gm ABCD = ||gm ABEF.

Theorem 3

Parallelograms on equal bases and between the same parallels are equal in area.



Hypothesis.—ABCD, EFGH are ||gms on the equal bases AB, EF and between the same ||s AF, DG.

To prove that ||gm ABCD = ||gm EFGH.

Construction.—Draw AK, BL, EM, FN each ⊥ to both AF, DG.

Proof.— ∵ AB = EF,

and AK = EM, (I—20, p. 67.)

∴ rect. KB = rect MF.

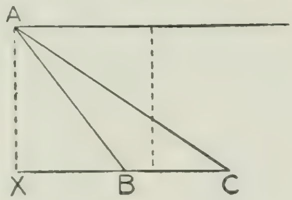
But ||gm ABCD = rect. KB, (II—1, p. 95.)

and ||gm EFGH = rect. MF,

∴ ||gm ABCD = ||gm EFGH.

74. Draw an acute-∠d △ ABC. Draw the ⊥ from A to BC. Draw through A, a st. line ∥ BC. Show that the ⊥ distance between these ∥ lines at any place = the altitude of △ ABC.

Draw an obtuse-∠d △ ABC, having the obtuse ∠ at B. Draw the altitude AX. Show that it falls without the △. Draw through A, a st. line ∥ BC. Show that the distance between these ∥ lines at any place = the altitude of the △.

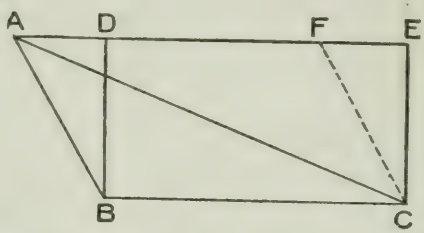


Taking C as the vertex and AB as the base, draw the altitude.

If a △ is between two ∥s, having its base in one of the ∥s and its vertex in the other, its altitude is the distance between the ∥s.

Theorem 4

The area of a triangle is half that of the rectangle on the same base and of the same altitude as the triangle.



Hypothesis.—**ABC** is a △ and **DBCE** a rectangle on the same base and of the same altitude **BD**.

To prove that area of △ **ABC** = half that of rect. **DBCE**.

Construction.—Through **C** draw **CF** ∥ **BA**.

Proof.—∵ **AC** is a diagonal of ∥gm **ABCF**,

∴ △ **ABC** = half of ∥gm **ABCF**. (I—20, p. 67.)

But ∥gm **ABCF** = rect. **DBCE**, (II.—1, p. 95.)

∴ △ **ABC** = half of rect. **DBCE**.

Cor.—If a be the measure of the base of a △ and b the measure of its altitude, the measure of its area is $\frac{1}{2}ab$.

75.—Practical Exercises

1. Draw a rt.-∠d △ having the sides that contain the right ∠ 56 mm. and 72 mm. Find the area of the △.

2. Make a △ **ABC**, having b = 6 cm., c = 8 cm., and ∠ **A** = 72°. Find its area.

3. Draw a △ having its sides 73 mm., 57 mm. and 48 mm. Find its area.

4. Find the area of the $\triangle$: $a = 10$ cm., $\angle$ **B** $= 42°$, $\angle$ **C** $= 58°$.

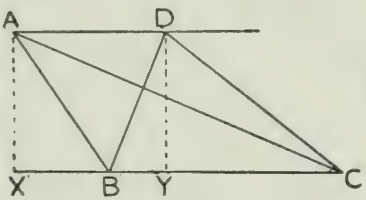
5. The sides of a triangular field are 36 chains, 25 chains and 29 chains. Draw a diagram and find the number of acres in the field. (Scale: 1 mm. to the chain.)

6. Two sides of a triangular field are 41 and 38 chains and the contained $\angle$ is 70°. Find its area in acres.

ANSWERS:—1. 20·16 sq. cm.; 2. 23 sq. cm. nearly; 3. 13·7 sq. cm. nearly; 4. 28·8 sq. cm.; 5. 36 ac.; 6. 73 ac. nearly.

THEOREM 5

Triangles on the same base and between the same parallels are equal in area.



Hypothesis.—**ABC, DBC** are $\triangle$s on the same base **BC** and between the same $\|$s **AD, BC.**

To prove that $\triangle$ **ABC** $=$ $\triangle$ **DBC.**

Construction.—Draw **AX, DY** $\perp$ **BC.**

Proof.— $\triangle$ **ABC** $= \frac{1}{2}$ rect. **AX . BC.** (II—4, p. 100.)

$\triangle$ **DBC** $= \frac{1}{2}$ rect. **DY . BC.**

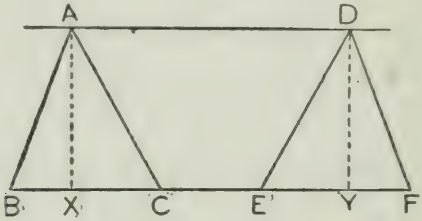
But, $\because$ **AX** $=$ **DY,**

$\therefore$ rect. **AX . BC** $=$ rect. **DY. BC.**

and $\therefore$ $\triangle$ **ABC** $=$ $\triangle$ **DBC.**

Theorem 6

Triangles on equal bases and between the same parallels are equal in area.



Hypothesis.—ABC, DEF are $\triangle$s on equal bases BC, EF and between the same $\|$s AD, BF.

To prove that $\triangle$ ABC $=$ $\triangle$ DEF.

Construction.—Draw AX, DY $\perp$ BF.

Proof.— $\triangle$ ABC $= \frac{1}{2}$ rect. AX . BC. (II.—4, p. 100.)

$\triangle$ DEF $= \frac{1}{2}$ rect. DY . EF.

But, $\because$ BC $=$ EF,

and AX $=$ DY, (I.—20, p. 67.)

$\therefore$ rect. AX . BC $=$ rect. DY . EF.

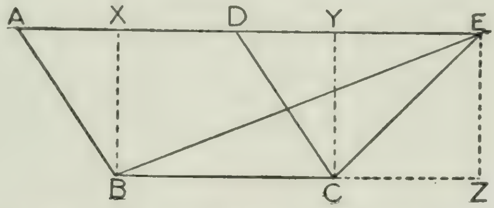
Hence, $\triangle$ ABC $=$ $\triangle$ DEF.

Cor. 1.—Triangles on equal bases and of the same altitude are equal in area.

Cor. 2.—A median bisects the area of the triangle.

THEOREM 7

If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double the triangle.



Hypothesis.—ABCD is a ||gm and EBC a △ on the same base BC and between the same ||s AE, BC.

To prove that ||gm ABCD = twice △ EBC.

Construction.—Draw BX, CY, EZ ⊥ BC and AE.

Proof.—||gm ABCD = rect. BX . BC. (II—1, p. 95.)

△ EBC = $\frac{1}{2}$ rect. EZ . BC. (II—4, p. 100.)

But, ∵ BX = EZ (I—20, p. 67.)

∴ rect. BX . BC = rect. EZ . BC.

And ∴ ||gm ABCD = twice △ EBC.

76.—Exercises

1. △s ABC, DEF are between the same ||s AD and BCEF, and BC > EF. Prove that △ ABC > △ DEF.

2. On the same base with a ||gm construct a rectangle equal in area to the ||gm.

3. On the same base with a given ||gm, construct a ||gm equal in area to the given ||gm, and having one of its sides equal to a given st. line.

4. Construct a rect. equal in area to a given ||gm, and having one of its sides equal to a given st. line.

5. Make a ||gm with sides 5 cm. and 3 cm., and contained ∠ 125°. Construct an equivalent rect. having one side 1·5 cm.

6. On the same base as a given △ construct a rect. equal in area to the △.

7. Construct a rect. equal in area to a given △, and having one of its sides equal to a given st. line.

8. On the same base with a ||gm construct a rhombus equal in area to the ||gm.

9. Construct a rhombus equal in area to a given ||gm, and having each of its sides equal to a given st. line.

10. On the same base with a given △, construct a rt.-∠d △ equal in area to the given △.

11. On the same base with a given △, construct an isosceles △ equal in area to the given △.

12. If, in the ||gm **ABCD**, **P** be any point between **AB**, **CD** produced indefinitely, the sum of the △s **PAB**, **PCD** equals half the ||gm; and if **P** be any point not between **AB**, **CD**, the difference of the △s **PAB**, **PCD** equals half the ||gm.

13. **AB** and **ECD** are two || st. lines; **BF**, **DF** are drawn || **AD**, **AE** respectively; prove that △s **ABC**, **DEF** are equal to each other.

14. On the same base with a given △, construct a △ equal in area to the given △, and having its vertex in a given st. line.

15. **If two △s have two sides of one respectively equal to two sides of the other and the contained ∠s supplementary, the △s are equal in area.**

16. **ABCD** is a ||gm, and **P** is a point in the diagonal **AC**. Prove that △ **PAB** = △ **PAD**.

17. **P** is a point within a ||gm **ABCD**. Prove that △ **PAC** equals the difference between △s **PAB, PAD.**

18. In △ **ABC, BC** and **CA** are produced to **P** and **Q** respectively, such that **CP** = one-half of **BC**, and **AQ** = one-half of **CA**. Show that △ **QCP** = three-fourths of △ **ABC.**

19. The medians **BE, CD** of the △ **ABC** intersect at **F.** Show that △ **BFC** = quadrilateral **ADFE.**

20. On the sides **AB, BC** of a △ the || gms **ABDE, CBFG** are described external to the △. **ED** and **GF** meet at **H** and **BH** is joined. On **AC** the ||gm **CAKL** is described with **CL** and **AK** || and = **HB.** Prove ||gm **AL** = ||gm **AD** + ||gm **CF.**

21. Two △s are equal in area and between the same ||s. Prove that they are on equal bases.

22. Of all △s on a given base and between the same ||s, the isosceles △ has the least perimeter.

23. **ABCD** is a ||gm, and **E** is a point such that **AE, CE** are respectively ⊥ and || to **BD.** Show that **BE** = **CD.**

24. The side **AB** of ||gm **ABCD** is produced to **E** and **DE** cuts **BC** at **F. AF** and **CE** are joined. Prove that △ **AFE** = △ **CBE.**

25. In the quadrilateral **ABCD, AB** || **CD.** If **AB** = a, **CD** = b and the distance between **AB** and **CD** = h, show that the area of **ABCD** = $\frac{1}{2} h (a + b)$.

26. Two sides **AB, AC** of a △ are given in length, find the ∠ **A** for which the area of the △ will be greatest.

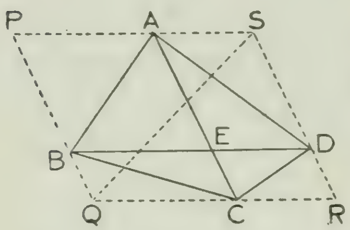
27. The medians **AD, BE** of △ **ABC** intersect at **G,** and **CG** is joined. Prove that the three lines **AG, BG, CG** trisect the area of the △.

28. Bisect the area of a △ by a st. line drawn through a vertex.

29. Trisect the area of a △ by two st. lines drawn through a vertex.

30. Bisect the area of a △ by a st. line drawn through a given point in one of the sides.

31. Trisect the area of a △ by two st. lines drawn through a given point in one of the sides.



32. The area of any quadrilateral **ABCD** is equal to that of a △ having two sides and their included ∠ respectively equal to the diagonals of the quadrilateral and their included ∠.

NOTE.—*Draw* **PS** *and* **QR** ∥ **BD, PQ** *and* **SR** ∥ **AC.** *Join* **SQ.**

33. Prove that in a rhombus the distance between one pair of opposite sides equals the distance between the other pair.

34. ∥gms are described on the same base and between the same ∥s. Find the locus of the intersection of their diagonals.

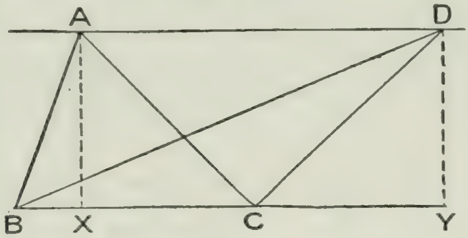
35. Prove that the area of a rhombus is half the product of the lengths of its diagonals.

36. **ABCD** is a quadrilateral in which **AB** ∥ **CD, E** is the middle point of **AD.** Prove that △ **BEC** = $\frac{1}{2}$ quadrilateral **ABCD.**

37. Divide a given △ into seven equal parts.

THEOREM 8

If two equal triangles are on the same side of a common base, the straight line joining their vertices is parallel to the common base.



Hypothesis.—ABC, DBC are two equal △s on the same side of the common base BC.

To prove that AD ∥ BC.

Construction.—Draw AX and DY ⊥ BC.

Proof.— △ ABC = ½ rect. BC . AX. (II—4, p. 100.)

△ DBC = ½ rect. BC . DY;

but △ ABC = △ DBC,

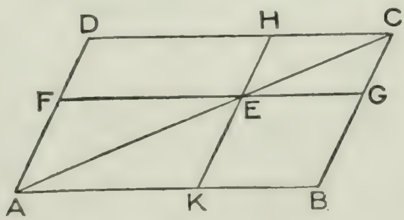
∴ ½ rect. BC . AX = ½ rect. BC . DY

and hence AX = DY,

that is, AX and DY are both = and ∥ to each other

∴ AD ∥ XY. (I—19, p. 66.)

77. If, through any point E, in the diagonal AC of a parallelogram BD, two straight lines FEG, HEK be drawn parallel respectively to the sides DC, DA of the parallelogram,

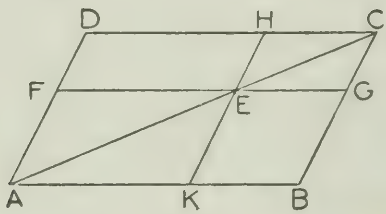


the ∥gms FK and HG are said to be parallelograms

about the diagonal AC, and the ||gms DE, EB are called the complements of the ||gms FK, HG, which are about the diagonal.

Theorem 9

The complements of the parallelograms about the diagonal of any parallelogram are equal to each other.



Hypothesis.—FK and HG are ||gms about the diagonal AC of the ||gm ABCD.

To prove that the complements DE, EB are equal to each other.

Proof.— $\because$ AE is a diagonal of ||gm FK,

$\therefore \triangle$ AFE $= \triangle$ AKE. (I—20, p. 67.)

Similarly $\triangle$ HEC $= \triangle$ EGC.

$\therefore \triangle$ AFE $+ \triangle$ HEC $= \triangle$ AKE $+ \triangle$ EGC.

But, $\because$ AC is a diagonal of ||gm ABCD

$\therefore \triangle$ ADC $= \triangle$ ABC.

$\therefore \triangle$ ADC $-$ ($\triangle$ AFE $+ \triangle$ HEC)

$= \triangle$ ABC $-$ ($\triangle$ AKE $+ \triangle$ EGC).

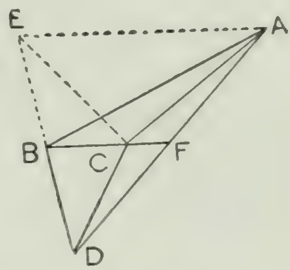
$\therefore$ ||gm DE $=$ ||gm EB.

78.—Exercises

1. **If two equal △s be on equal segments of the same st. line and on the same side of the line, the st. line joining their vertices is ∥ to the line containing their bases.**

2. Through P, a point within the ∥gm ABCD, EPF is drawn ∥ AB and GPH is drawn ∥ AD. If ∥gm AP = ∥gm PC, show that P is on the diagonal BD. (Converse of Theorem 9.)

3. Two equal △s ABC, DBC are on opposite sides of the same base. Prove that AD is bisected by BC, or BC produced.



Note.—*Produce* DB *making* BE = DB. *Join* EA, EC.

Give another proof of this proposition using ⊥s from A and D to BC and II—4, p. 100.

4. **The median drawn to the base of a △ bisects all st. lines drawn ∥ to the base and terminated by the sides, or the sides produced.**

5. P is a point within a △ ABC and is such that △ PAB + △ PBC is constant. Prove that the locus of P is a st. line ∥ AC.

6. ∥gms about the diagonal of a square are squares.

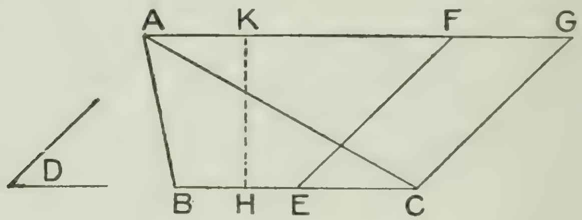
7. D, E, F are respectively the middle points of the sides BC, CA, AB in the △ ABC. Prove △ BEF = △ CEF and hence that EF ∥ BC.

8. In the diagram of II—9, show that FK ∥ HG.

Constructions

Problem 1

To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.



Let **ABC** be the given △ and **D** the given ∠.

It is required to construct a ||gm equal in area to △ **ABC** and having one ∠ equal to ∠ **D**.

Construction.—Through **A** draw **AF** || **BC**. Bisect **BC** at **E.** At **E** make ∠ **CEF** = ∠ **D.** Through **C** draw **CG** || **EF.**

FC is the required ||gm.

Proof.—Draw any line **HK** ⊥ to the two || st. lines.

HK is the common altitude of the ||gm **FC** and the △ **ABC.**

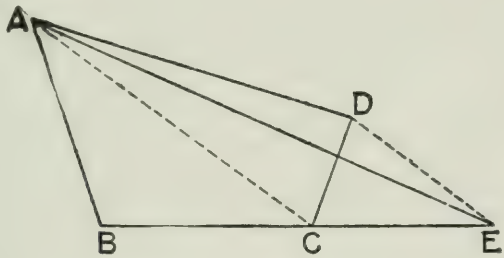
||gm **FC** = rect. **EC . HK.** (II—1, p. 95.)

= $\frac{1}{2}$ rect. **BC . HK,** ∵ **EC** = $\frac{1}{2}$ **BC,**

= △ **ABC.** (II—4, p. 100.)

Problem 2

To construct a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to construct a $\triangle$ equal in area to ABCD.

Construction.—Join AC. Through D draw DE $\parallel$ AC and meeting BC produced at E. Join AE.

$$\triangle \text{ ABE} = \text{quadrilateral ABCD.}$$

Proof.—$\because$ DE $\parallel$ AC,

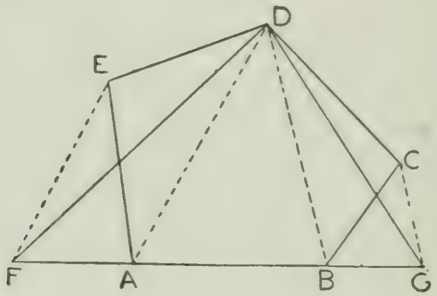
$\therefore$ $\triangle$ EAC = $\triangle$ DAC. (II—5, p. 101.)

To each of these equals add $\triangle$ ABC.

Then $\triangle$ ABE = quadrilateral ABCD.

Problem 3

To construct a triangle equal in area to a given rectilineal figure.



Let the pentagon ABCDE be the given rectilineal figure.

Construction.—Join AD, BD. Through E, draw EF ∥ AD and meeting BA at F. Through C draw CG ∥ BD and meeting AB at G.

Join DF, DG.

$\triangle$ DFG = figure ABCDE.

Proof.— ∵ EF ∥ AD,

∴ $\triangle$ DFA = $\triangle$ DEA. (II—5, p. 101.)

∵ CG ∥ DB,

∴ $\triangle$ DGB = $\triangle$ DCB.

∴ $\triangle$ DFA + $\triangle$ DAB + $\triangle$ DBG

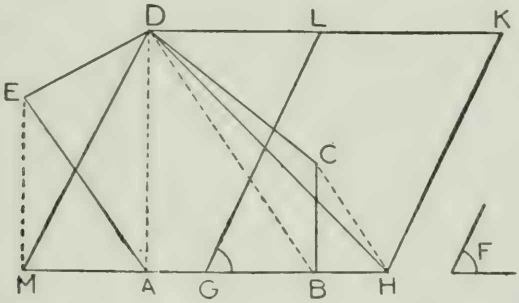
= $\triangle$ DEA + $\triangle$ DAB + $\triangle$ DCB;

i.e., $\triangle$ DFG = figure ABCDE.

By this method a $\triangle$ may be constructed equal in area to a given rectilineal figure of any number of sides; *e.g.*, for a figure of seven sides, an equivalent figure of five sides may be constructed, and then, as in the construction just given, a $\triangle$ may be constructed equal to the figure of five sides.

PROBLEM 4

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCDE be the given rectilineal figure and F the given $\angle$.

It is required to construct a ||gm = ABCDE, and having an $\angle = \angle$ F.

Construction.—Make $\triangle$ DMH equal in area to figure ABCDE. (II—Prob. 3, p. 112.)

Make ||gm LGHK = $\triangle$ DMH, and having $\angle$ LGH = $\angle$ F. (II—Prob. 1, p. 110.)

Then ||gm LGHK = figure ABCDE, and has $\angle$ LGH = $\angle$ F.

79.—Exercises

1. Construct a rect. equal in area to a given $\triangle$.

2. Construct a rect. equal in area to a given quadrilateral.

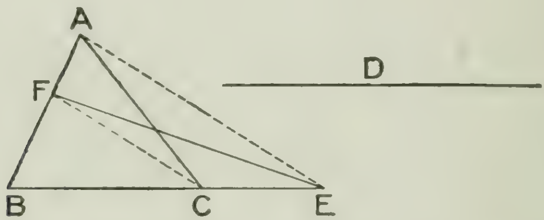
3. Construct a quadrilateral equal in area to a given hexagon.

4. On one side of a given $\triangle$ construct a rhombus equal in area to the given $\triangle$.

5. Construct a $\triangle$ equal in area to a given || gm, and having one of its $\angle$s = a given $\angle$.

Problem 5

To construct a triangle equal in area to a given triangle and having one of its sides equal to a given straight line.



Let ABC be the given $\triangle$ and D the given st. line.

It is required to make a $\triangle = \triangle$ ABC and having one side = D.

Construction.—From BC, produced if necessary, cut off BE = D. Join AE. Through C draw CF $\parallel$ EA and meeting BA, or BA produced at F. Join FE.

FBE is the required $\triangle$.

Proof.— $\because$ FC $\parallel$ AE,

$\therefore$ $\triangle$ FCE = $\triangle$ AFC. (II—5, p. 101.)

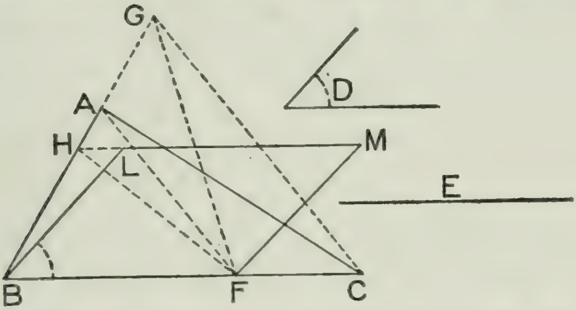
$\therefore$ $\triangle$ FBC + $\triangle$ FCE = $\triangle$ FBC + $\triangle$ AFC,

i.e., $\triangle$ FBC = $\triangle$ ABC,

and side BE was made = D.

Problem 6

On a straight line of given length to make a parallelogram equal in area to a given triangle and having an angle equal to a given angle.



Let ABC be the given △, E the given st. line and D the given ∠.

It is required to make a ‖gm equal in area to △ ABC, having one side equal in length to E, and one ∠ equal to D.

Construction.—From BC, produced if necessary, cut off BF = E. Join AF. Through C draw CG ‖ FA meeting BA, or BA produced, at G. Join GF. Bisect BG at H. Through H draw HM ‖ BC. At B make ∠ CBL = ∠ D. Through F draw FM ‖ BL.

LBFM is the required ‖gm.

Proof.—Join HF.

△s GAF, AFC are on the same base AF and have the same altitude, ∴ they are equal. (II—5, p. 101.)

To each of these equal △s add the △ ABF, and

△ GBF = △ ABC.

△ GBF = twice △ HBF, (II—6, Cor. 2, p. 102.)

= ‖gm LBFM, (II—7, p. 103.)

∴ ‖gm LBFM = △ ABC.

Also ∠ LBF = ∠ D and side BF = E.

Areas of Squares

80.—A rectangle is said to be contained by two st. lines when its length is equal to one of the st. lines, and its breadth is equal to the other.

The symbol AB^2 should be read:—"the square on AB," and not "AB squared."

Theorem 10

The square on the sum of two straight lines equals the sum of the squares on the two straight lines increased by twice the rectangle contained by the straight lines.

Hypothesis.—AB, BC are the two st. lines placed in the same st. line so that AC is their sum.

To prove that

$$AC^2 = AB^2 + BC^2 + 2 \,.\, AB.BC.$$

Algebraic Proof

Proof.—Let a, b represent the number of units of length in AB, BC respectively.

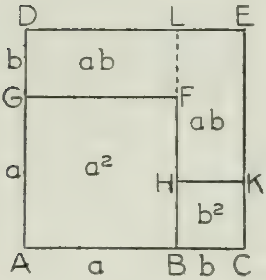
Area of the square on AC

$= (a + b)^2$

$= a^2 + b^2 + 2\,ab$

= area of square on AB + area of square on BC + twice the area of the rectangle contained by AB, and BC.

Geometric Proof



Construction.—On AC, AB, BC draw squares ACED, ABFG, BCKH. Produce BF to meet DE at L.

Proof.—

$$GD = AD - AG = AC - AB = BC, \text{ and } GF = AB.$$

$$\therefore GL = \text{rect. } AB.BC.$$

$$KE = CE - CK = AC - BC = AB, \text{ and } HK = BC.$$

$$\therefore HE = \text{rect. } AB.BC.$$

$$\begin{aligned} AC^2 &= AE \\ &= AF + BK + GL + HE \\ &= AB^2 + BC^2 + 2\ AB.BC. \end{aligned}$$

Theorem 11

The square on the difference of two straight lines equals the sum of the squares on the two straight lines diminished by twice the rectangle contained by the straight lines.

Hypothesis.—**AB**, **BC** are two st. lines, of which **AB** is the greater, placed in the same st. line, and so that **AC** is their difference.

To prove that

$$AC^2 = AB^2 + BC^2 - 2 . AB . BC.$$

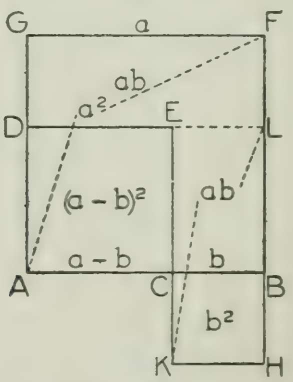
Algebraic Proof

Proof.—Let a, b represent the number of units of length in **AB**, **BC** respectively.

$$\text{Area of square on AC} = (a - b)^2$$
$$= a^2 + b^2 - 2ab$$

= the sum of the squares on **AB** and **BC** diminished by twice the area of the rectangle contained by **AB** and **BC**.

Geometric Proof



Construction.—On **AC, AB, BC** draw the squares **ACED, ABFG, BCKH.** Produce **DE** to meet **BF** at **L.**

Proof.—DG = AG − AD = AB − AC = BC, and DL = AB.

$\therefore$ DF = rect. AB . BC.

KE = KC + CE = BC + AC = AB, and KH = BC.

$\therefore$ KL = rect. AB . BC.

$$
\begin{aligned}
AC^2 &= AE \\
&= AF + KB - (DF + KL) \\
&= AB^2 + BC^2 - 2\,AB\,.\,BC.
\end{aligned}
$$

Theorem 12

The difference of the squares on two straight lines equals the rectangle of which the length is the sum of the straight lines and the breadth is the difference of the straight lines.

A, B are two st. lines, of which A > B.

To prove that the square on A diminished by the square on B = the rect. contained by A + B and A − B.

Proof.—Let a, b represent the number of units in A and B respectively.

The difference of the squares on A and B

$$
\begin{aligned}
&= a^2 - b^2 \\
&= (a + b)\,(a - b) \\
&= \text{the area of the rectangle}
\end{aligned}
$$

contained by A + B and A − B.

81.—Exercises

1. Draw a diagram illustrative of Theorem 12.

2. The square on the sum of three st. lines equals the sum of the squares on the three st. lines increased by twice the sum of the rectangles contained by each pair of the st. lines.

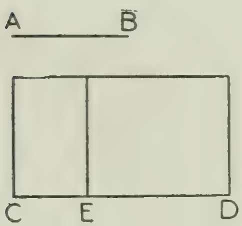
Illustrate by diagram.

3. The sum of the squares on two unequal st. lines > twice the rectangle contained by the two st. lines.

4. The sum of the squares on three unequal st. lines > the sum of the rectangles contained by each pair of the st. lines.

5. Construct a rectangle equal to the difference of two given squares.

6. If there be two st. lines **AB** and **CD**, and **CD** be divided at **E** into any two parts, the rect. **AB.CD** = rect. **AB.CE** + rect. **AB.ED**.



Let **AB** = p units of length

CE = q " " "

ED = r " " "

Area of **AB.CD** = $p\ (q + r)$

" " **AB.CE** = pq

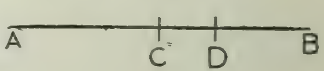
" " **AB.ED** = pr.

But $p\ (q+r) = pq + pr$.

∴ **AB.CD** = **AB.CE** + **AB.ED**.

7. Give a diagram illustrating the identity $(a + b)(c + d) = ac + ad + bc + bd$, taking a, b, c, d to be respectively the number of units in four st. lines.

8. **C** is the middle point of a st. line **AB**, and **D** is any other point in the line. Prove:

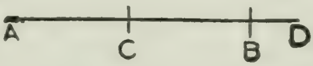


(1) $AD.DB = AC^2 - CD^2$;

(2) $AD^2 + DB^2 = 2\ AC^2 + 2\ CD^2$.

(Let $AC = CB = p$, $CD = q$).

9. **C** is the middle point of a st. line **AB**, and **D** is any point in **AB** produced. Prove:



(1) $AD.DB = CD^2 - AC^2$;

(2) $AD^2 + DB^2 = 2\ AC^2 + 2\ CD^2$.

10. Draw diagrams to illustrate the four results in exercises 8 and 9.

11. Draw a diagram illustrating the identity $(a + b)^2 - (a - b)^2 = 4\ ab$.

12. If **A**, **B**, **C**, **D** be four points in order in a st. line, $AB.CD + AD.BC = AC.BD$.

Illustrate by a diagram.

13. **AB** is a st. line in which **C** is any point. Prove that $AB^2 = AB.AC + AB.CB$.

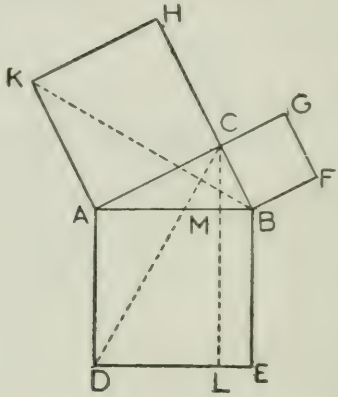
14. Construct a △ having two sides and the median drawn to one of these sides equal to three given st. lines.

15. Construct a △ having two sides and the median drawn to the third side equal to three given st. lines.

16. In a given ||gm inscribe a rhombus having one vertex at a given point in a side of the ||gm.

THEOREM 13

The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.



Hypothesis.—ABC is a $\triangle$ in which $\angle$ ACB is a rt. $\angle$, and AE, BG, CK are squares on AB, BC and CA.

To prove that $AB^2 = AC^2 + BC^2$.

Construction.—Through C draw CL $\parallel$ AD.

Join KB. CD.

Proof.—$\because$ $\angle$s HCA, ACB, BCG are rt. $\angle$s,

$\therefore$ $\angle$s HCB, ACG are st. $\angle$s.

and $\therefore$ HCB, ACG are st. lines.

$\angle$ BAD = $\angle$ KAC,

to each add $\angle$ CAB,

then $\angle$ CAD = $\angle$ KAB.

In $\triangle$s CAD, KAB, $\left\{\begin{array}{c} CA = KA \\ AD = AB \\ \angle CAD = \angle KAB \end{array}\right.$

$\therefore \triangle CAD = \triangle KAB$ (I—2, p. 16.)

$\because$ rect. ADLM and $\triangle$ CAD are on the same base AD and between the same ||s CL, AD,

$\therefore$ rect. AL = twice $\triangle$ CAD. (II—7, p. 103.)

Similarly, sq. HA = twice $\triangle$ KAB.

$\therefore$ rect. AL = sq. HA.

In the same manner, by joining CE and AF, it may be shown that

rect. BL = sq. BG.

$\therefore$ rect. AL + rect. BL = sq. HA + sq. BG,

i.e., $AB^2 = AC^2 + BC^2$.

82. Many proofs have been given for this important theorem. Pythagoras (570 to 500 B.C.) is said by tradition to have been the first to prove it, and from that it is commonly called the Theorem of Pythagoras, or the Pythagorean Theorem. The proof given above is attributed to Euclid (about 300 B.C.). An alternative proof is given in Book IV.

83.—Exercises

1. Draw two st. lines 5 cm. and 6 cm. in length. Describe squares on both, and make a square equal in area to the two squares. Measure the side of this last square and check your result by calculation.

2. Draw three squares having sides 1 in., 2 in. and $2\frac{1}{2}$ in. Make one square equal to the sum of the three. Check by calculation.

3. Draw two squares having sides $1\frac{1}{2}$ in. and $2\frac{1}{2}$ in. Make a third square equal to the difference of the first two. Check by calculation.

4. Draw two squares having sides 9 cm. and 6 cm. Make a third square equal to the difference of the first two. Check your result by calculation.

5. Draw any square and one of its diagonals. Draw a square on the diagonal and show that it is double the first square.

6. Draw a square having each side 4 cm. Draw a second square double the first. Measure a side, and check by calculation.

7. Draw a square having one side 45 mm. Draw a second square three times the first. Measure its side, and check by calculation.

8. Draw three lines in the ratio 1:2:3. Draw squares on the lines, and divide the two larger so as to show that the squares are in the ratio 1:4:9.

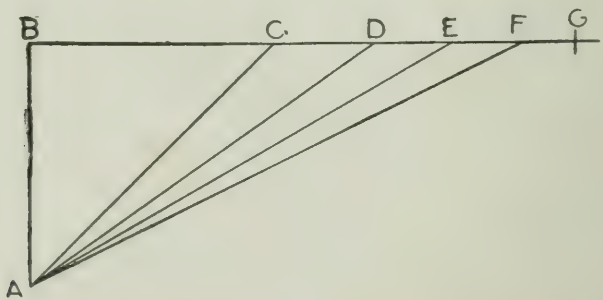
9. Draw a st. line $\sqrt{2}$ in. in length.

10. Draw a st. line $\sqrt{3}$ in. in length.

11. Draw a st. line $\sqrt{5}$ in. in length.

12. Draw any rt.-$\angle$d $\triangle$. Describe equilateral $\triangle$s on the three sides. Find the areas of the $\triangle$s and compare that on the hypotenuse with the sum of those on the other two sides.

13.



AB is one inch in length, $\angle$ **B** a rt. $\angle$, **BC** is one inch **BD** is cut off = **AC**, **BE** = **AD**, **BF** = **AE**, **BG** = **AF**, etc. Show that **BD** = $\sqrt{2}$ in., **BE** = $\sqrt{3}$ in., **BF** = $\sqrt{4}$ = 2 in., **BG** = $\sqrt{5}$ in., etc.

14. Construct a square equal to half a given square.

15. **If a $\perp$ be drawn from the vertex of a $\triangle$ to the base, the difference of the squares on the segments of the base = the difference of the squares on the other two sides.**

Hence, prove that the altitudes of a $\triangle$ pass through one point.

16. **A** is a given st. line. Find another st. line **B**, such that the difference of the square on **A** and **B** may be equal to the difference of two given squares.

17. If the diagonals of a quadrilateral cut at rt. $\angle$s, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

18. The sum of the squares on the diagonals of a rhombus equals the sum of the squares on the four sides.

19. Five times the square on the hypotenuse of a rt.-$\angle$d $\triangle$ equals four times the sum of the squares on the medians drawn to the other two sides.

20. In an isosceles rt.-$\angle$d $\triangle$ the sides have the ratios $1:1:\sqrt{2}$.

21. If the angles of a $\triangle$ be 90°, 30°, 60°, the sides have the ratios $2:1:\sqrt{3}$.

22. Divide a st. line into two parts such that the sum of the squares on the parts equals the square on another given st. line. When is this impossible?

23. In the st. line **AB** produced find a point **C** such that the sum of the squares on **AC, BC** equals the square on a given st. line.

24. Divide a given st. line into two parts such that the square on one part is double the square on the other part.

25. **ABCD** is a rect., and **P** is any point. Show that $PA^2 + PC^2 = PB^2 + PD^2$.

26. **ABC** is a $\triangle$ rt.-$\angle$d at **A**. **E** is a point on **AC** and **F** is a point on **AB**. Show that $BE^2 + CF^2 = EF^2 + BC^2$.

27. If two rt.-∠d △s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the △s are congruent.

28. The square on the side opposite an acute ∠ of a △ is less than the sum of the squares on the other two sides.

29. The square on the side opposite an obtuse ∠ of a △ is greater than the sum of the squares on the other two sides.

30. Construct a square that contains 20 square inches.

31. In the diagram of II—13, show that **KB**, **CD** cut at rt. ∠s.

32. In the diagram of II—13, if **KD** be joined, show that △ **KAD** = △ **ABC**.

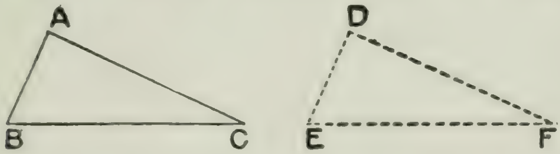
33. In the diagram of II—13, the distance of **E** from **AC** = **AC** + **CB**.

34. **ABC** is an isosceles rt.-∠d △ in which **C** is the rt. ∠. **CB** is produced to **D** making **BD** = **CB**. ⊥s to **AB**, **BD** at **A**, **D** respectively meet at **E**. Prove that **AE** = 2 **AB**.

Theorem 14

(Converse of Theorem 13)

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle.



Hypothesis.—ABC is a $\triangle$ in which $BC^2 = AB^2 + AC^2$.

To prove that $\angle$ A is a rt. $\angle$.

Construction.—Make a rt. $\angle$ D and cut off DE = AB, DF = AC.

Join EF.

$$BC^2 = AB^2 + AC^2 \qquad (Hyp.)$$
$$= DE^2 + DF^2$$
$$= EF^2 \ (\because D \text{ is a rt. } \angle). \qquad \text{(II—13, p. 122.)}$$

$\therefore$ BC = EF.

In $\triangle$s ABC, DEF, $\begin{cases} AB = DE, \\ AC = DF, \\ BC = EF, \end{cases}$

$$\therefore \angle A = \angle D. \qquad \text{(I—4, p. 22.)}$$

$\therefore$ $\angle$ A is a rt. $\angle$.

84.—Exercises

1. The sides of a △ are 3 in., 4 in. and 5 in. Prove that it is a rt.-∠d △.

2. The sides of a △ are 13 mm., 84 mm. and 85 mm. Prove that it is a rt.-∠d △.

3. In the quadrilateral ABCD, $AB^2 + CD^2 = BC^2 + AD^2$. Prove that the diagonals AC, BD cut at rt. ∠s.

4. If the sq. on one side of a △ be less than the sum of the squares on the other two sides, the ∠ contained by these sides is an acute ∠. (Converse of § 83, Ex. 28.)

5. State and prove a converse of § 83, Ex. 29.

6. Using a tape-measure, or a knotted cord, and Ex. 1, draw a st. line at rt. ∠s to a given st. line.

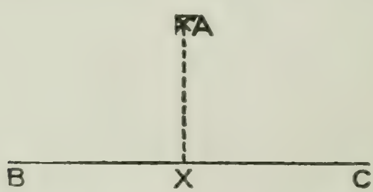
7. Show that, if the sides of a △ are represented by $m^2 + n^2$, $m^2 - n^2$, $2\,mn$, where m and n are any numbers, the △ is rt.-∠d.

Use this result to find numbers representing the sides of a rt.-∠d △.

---

85. **Definition.**—If a perpendicular be drawn from a given point to a given straight line, the foot of the perpendicular is said to be the **projection of the point on the line.**

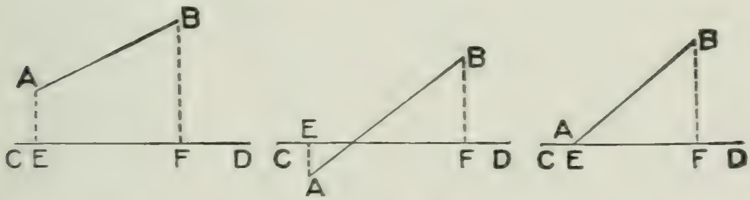
From the point A the ⊥ AX is drawn to the line BC.



The point X is the projection of the point A on the st. line BC.

86. **Definition.**—If from the ends of a given straight line perpendiculars be drawn to another given straight line, the segment intercepted on the second straight line is called the **projection of the first straight line on the second straight line.**

AB is a st. line of fixed length and **CD** another st. line. **AE**, **BF** are drawn $\perp$ **CD**.



EF is the projection of **AB** on **CD**.

87.—Exercises

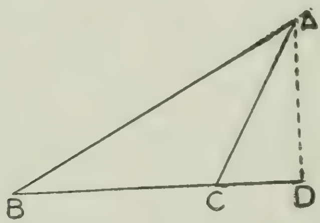
1. Show that a st. line of fixed length is never less than its projection on another st. line. In what case are they equal? In what case is the projection of one st. line on another st. line just a point?

2. **ABC** is a $\triangle$ having a = 36 mm., b = 40 mm. and c = 45 mm. Draw the $\triangle$ and measure the projection of **AB** on **BC**. (*Ans.* 23·9 mm. nearly.)

3. **ABC** is a $\triangle$ having a = 5 cm., b = 7 cm., c = 10 cm. Draw the $\triangle$ and measure the projection of **AB** on **BC**. (*Ans.* 76 mm.)

Theorem 15

In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides that contain the obtuse angle increased by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.—ABC is a △ in which ∠ C is obtuse, and CD is the projection of CA on CB.

To prove that $AB^2 = AC^2 + BC^2 + 2\ BC.CD$.

Proof.— ∵ ADB is a rt. ∠,

$\therefore AB^2 = BD^2 + AD^2$. (II—13, p. 122.)

∵ $BD = BC + CD$,

$\therefore BD^2 = BC^2 + CD^2 + 2\ BC.CD$. (II—10, p. 116.)

$\therefore AB^2 = BC^2 + CD^2 + 2\ BC.CD + AD^2$.

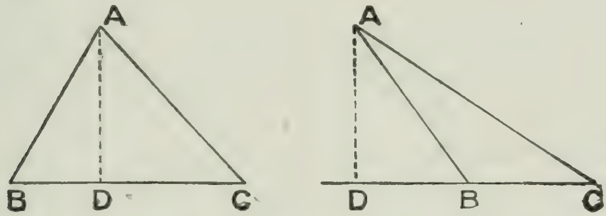
But ∵ ADC is a rt. ∠,

$\therefore CD^2 + AD^2 = AC^2$.

$\therefore AB^2 = AC^2 + BC^2 + 2\ BC.CD$.

Theorem 16

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the sides which contain the acute angle diminished by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.—ABC is a $\triangle$ in which $\angle$ C is acute, and CD is the projection of CA on CB.

To prove that $AB^2 = AC^2 + BC^2 - 2\ BC.CD$.

Proof.—$\because$ ADB is a rt. $\angle$,

$\therefore\ AB^2 = BD^2 + AD^2$. (II—13, p. 122.)

$\because$ BD is the difference between BC and CD,

$\therefore\ BD^2 = CD^2 + BC^2 - 2\ BC.CD$. (II—11, p. 118.)

$\therefore\ AB^2 = CD^2 + BC^2 - 2\ BC.CD + AD^2$.

But, $\because$ ADC is a rt. $\angle$,

$\therefore\ CD^2 + AD^2 = AC^2$.

$\therefore\ AB^2 = AC^2 + BC^2 - 2\ BC.CD$.

88.—Exercises

1 ABC is a $\triangle$ having C an $\angle$ of 60°. Show that sq. on AB = sq. on BC + sq. on AC − rect. BC.AC.

2. ABC is a $\triangle$ having C an $\angle$ of 120°. Show that sq. on AB = sq. on BC + sq. on AC + rect. BC.AC.

3. **ABC** is a △, **CD** the projection of **CA** on **CB**, and **CE** the projection of **CB** on **CA**. Show that rect. **BC.CD** = rect. **AC.CE**.

4. **In any △ the sum of the squares on two sides equals twice the square on half the base together with twice the square on the median drawn to the base.**

NOTE.—*Draw a ⊥ from the vertex to the base, and use II—15 and II—16.*

5. In any quadrilateral the sum of the squares on the four sides exceeds the sum of the squares on the diagonals by four times the square on the st. line joining the middle points of the diagonals.

What does this proposition become when the quadrilateral is a ||gm?

6. **ABC** is a △ having a = 47 mm., b = 62 mm., and c = 84 mm. **D, E, F** are the middle points of **BC, CA, AB** respectively. Calculate the lengths of **AD, BE** and **CF**. Test your results by drawing and measurement.

7. The squares on the diagonals of a quadrilateral are together double the sum of the squares on the st. lines joining the middle points of opposite sides.

8. If the medians of a △ intersect at **G**,

$$AB^2 + BC^2 + CA^2 = 3\,(GA^2 + GB^2 + GC^2).$$

9. **C** is the middle point of a st. line **AB**. **P** is any point on the circumference of a circle of which **C** is the centre. Show that $PA^2 + PB^2$ is constant.

10. Two circles have the same centre. Prove that the sum of the squares of the distances from any point on the circumference of either circle to the ends of the diameter of the other is constant.

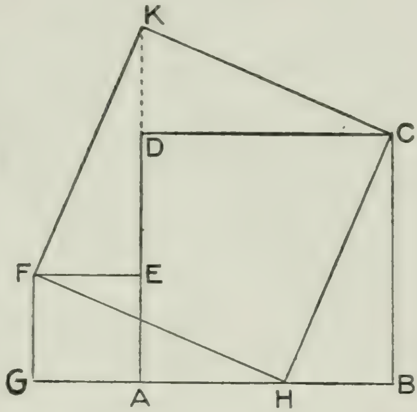
11. The square on the base of an isosceles △ is equal to twice the rect. contained by either of the equal sides and the projection on it of the base.

12. Prove II—13 from the following construction:—

Draw two squares, ABCD, AEFG, having AD, AE in the same st. line.

Cut off GH and EK each = AB.

Join FH, HC, CK, KF.



13. If two sides of a △ be unequal, the median drawn to the shorter side is greater than the median drawn to the longer side.

14. If, from any point P within △ ABC, ⊥s PX, PY, PZ be drawn to BC, CA, AB respectively,

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2.$$

15. D, E, F are the middle points of BC, CA, AB respectively in △ ABC. Prove that

$$3(AB^2 + BC^2 + CA^2) = 4(AD^2 + BE^2 + CF^2).$$

16. G is the centroid of △ ABC, and P is any point. Show that

$$PA^2 + PB^2 + PC^2 = AG^2 + BG^2 + CG^2 + 3\,PG^2.$$

17. Find the point P in the plane of the △ ABC such that the sum of the squares on PA, PB, PC may be the least possible.

18. Check the results in Exs. 2 and 3, § 87, by calculation.

19. If, in II—15, the obtuse ∠ becomes greater and greater and finally becomes a st. ∠, what does the theorem become?

20. If, in the diagram of II—16, the ∠ C becomes more and more acute and finally the point A comes down to the line BC, what does the theorem become?

Miscellaneous Exercises

1. If a quadrilateral be bisected by each of its diagonals, it is a ||gm.

2. If any point **P** in the diagonal **AC** of the ||gm **ABCD** be joined to **B** and **D**, the ||gm is divided into two pairs of equal $\triangle$s.

3. The diagonals of a ||gm divide the ||gm into four equal parts.

4. If two sides of a quadrilateral are || to each other, the st. line joining their middle points bisects the area of the quadrilateral.

5. If two sides of a quadrilateral are || to each other, the st. line joining their middle points passes through the intersection of the diagonals.

6. If **P** is any point in the side **AB** of ||gm **ABCD**, and **PC**, **PD** are joined,

$$\triangle \text{PAD} + \triangle \text{PBC} = \triangle \text{PDC}.$$

7. Prove that the following method of bisecting a quadrilateral by a st. line drawn through one of its vertices is correct:—Let **ABCD** be the quadrilateral. Join **AC**, **BD**. Bisect **BD** at **E**. Through **E** draw **EF** || **AC** and meeting **BC**, or **CD**, at **F**. Join **AF**. **AF** bisects the quadrilateral.

NOTE.—*Join* **AE**, *and* **EC**.

8. If the diagonals of ||gm **ABCD** cut at **O**, and **P** is any point within the $\triangle$ **AOB**, $\triangle$ **CPD** = $\triangle$ **APB** + $\triangle$ **APC** + $\triangle$ **BPD**.

NOTE.—*Join* **PO**.

9. **ABC** is an isosceles $\triangle$ having **AB** = **AC**, and **D** is a point in the base **BC**, or **BC** produced. Prove that the difference between the squares on **AD** and **AC** = rect. **BD.DC**.

10. P, Q, R, S are respectively the middle points of the sides AB, BC, CD, DA in the quadrilateral ABCD. Prove that $AB^2 + CD^2 + 2\,PR^2 = CB^2 + DA^2 + 2\,QS^2$.

11. $BY \perp AC$ and $CZ \perp AB$ in $\triangle$ ABC. Prove that BC^2 = rect. AB.BZ + rect. AC.CY.

12. L, M, N are three given points, and PQ a given st. line. Construct a rhombus ABCD, having its angular points A, C lying on the line PQ, and its three sides AB, BC, CD (produced if necessary) passing through L, M, N respectively.

13. Through D the middle point of the side BC of $\triangle$ ABC a st. line XDY is drawn cutting AB at X and AC produced through C at Y. Prove $\triangle$ AXY $>$ $\triangle$ ABC.

14. From the vertex A of $\triangle$ ABC draw a st. line terminated in BC and equal to the average of AB and AC.

15. AB and CD are two equal st. lines that are not in the same st. line. Find a point P such that $\triangle$ PAB $\equiv$ $\triangle$ PCD.

Show that, in general, two such points may be found.

16. EF drawn $\parallel$ to the diagonal AC of $\parallel$gm ABCD meets AD, DC, or those sides produced, in E, F respectively. Prove that $\triangle$ ABE $=$ $\triangle$ BCF.

17. Construct a rect. equal to a given square and such that one side equals a given st. line.

18. Find a point in one of two given intersecting st. lines such that the perpendiculars drawn from it to both the given lines may cut off from the other a segment of given length.

19. In the diagram of II—9, if BD, BE and DE be drawn, $\parallel$gm FK $-$ $\parallel$gm HG $=$ 2 $\triangle$ EBD.

20. ABC is an isosceles $\triangle$ in which C is a rt. $\angle$, and the bisector of $\angle$ A meets BC at D. Prove that CD = AB $-$ AC.

21. Place a st. line of given length between two given st. lines so as to be || a given st. line.

22. Describe a $\triangle$ = a given ||gm and such that its base = a given st. line, and one $\angle$ at the base = a given $\angle$.

23. Construct a ||gm equal and equiangular to a given ||gm, and such that one side is equal to a given st. line.

24. Construct a ||gm equal and equiangular to a given ||gm, and such that its altitude is equal to a given st. line.

25. **ABCD** is a quadrilateral. On **BC** as base construct a ||gm equal in area to **ABCD**, and having one side along **BA**.

26. Squares **ABDE**, **ACFG** have a common $\angle$ **A**, and **A**, **B**, **C** are in the same st. line. **AH** is drawn $\perp$ **BG** and produced to cut **CE** at **K**. Prove that **EK** = **KC**.

27. Make a rhombus **ABCD** in which $\angle$ **A** = 100°. A circle described with centre **A** and radius **AB** cuts **BC**, **CD** at **E**, **F** respectively. Prove that **AEF** is an equilateral $\triangle$.

28. A st. line **AB** is bisected at **C** and divided into two unequal parts at **D**. Prove that $AD^2 + DB^2 = 2\,AD \cdot DB + 4\,CD^2$.

29. **ABCD** is a quadrilateral in which **AB** || **CD**. Prove that

$$AC^2 + BD^2 = AD^2 + BC^2 + 2\,AB \cdot CD.$$

30. Trisect a given ||gm by st. lines drawn through one of its angular points.

31. The base **BC** of the $\triangle$ **ABC** is trisected at **D**, **E**. Prove that

$$AB^2 + AC^2 = AD^2 + AE^2 + 4\,DE^2.$$

32. **ACB**, **ADB** are two rt.-$\angle$d $\triangle$s on the same side of the same hypotenuse **AB**, and **AX**, **BY** are $\perp$ **CD** produced. Prove that

$$XC^2 + CY^2 = XD^2 + DY^2.$$

33. ABC is an isosceles $\triangle$, and XY is $\parallel$ BC and terminated in AB, AC. Prove

$$BY^2 = CY^2 + BC.XY.$$

34. Any rect. = half the rect. contained by the diagonals of the squares on two of its adjacent sides.

35. ABCD is a $\parallel$gm in which BD = AB. Prove that $BD^2 + 2\,BC^2 = AC^2$.

36. A rect. BDEC is described on the side BC of a $\triangle$ ABC. Prove that

$$AB^2 + AE^2 = AC^2 + AD^2.$$

37. BE, CD are squares described externally on the sides AB, AC of a $\triangle$ ABC. Prove that

$$BC^2 + ED^2 = 2\,(AB^2 + AC^2).$$

NOTE.—*Draw* EX, CY $\perp$ DA, AB *respectively, and rotate* $\triangle$ ABC *to the position in which* AB *coincides with* AE.

38. ABC is a $\triangle$ in which AX $\perp$ BC, and D is the middle point of BC. Prove that the difference of the squares on AB, AC = 2 BC.DX.

39. BC is the greatest and AB the least side in $\triangle$ ABC. D, E, F are the middle points of BC, CA, AB respectively; and X, Y, Z are the feet of the $\perp$s from A, B, C to the opposite sides. Prove that CA.EY = AB.FZ + BC.DX.

40. ABCD is a rect. in which E is any point in BC and F is any point in CD. Prove that ABCD = 2 $\triangle$ AEF + BE.DF.

41. A and B are two fixed points. Find the position of a point P such that $PA^2 + PB^2$ may be the least possible.

42. From a given point A draw three st. lines AB, AC, AD respectively equal to three given st. lines, and such that B, C, D are in the same st. line and BC = CD.

43. Find the locus of a point such that the sum of the squares on its distances from two given points is constant.

44. Find the locus of a point such that the difference of the squares on its distances from two given points is constant.

45. ABCD is a ||gm, P any point in BC, and Q any point in AP. Prove that △ BQC = △ PQD.

46. ABCD is a quadrilateral having AB ∥ CD, and AB + CD = BC. Prove that the bisectors of ∠s B and C intersect on AD.

47. ABC is a △ in which ∠ A is a rt. ∠, and AB > AC. Squares BCDE, CAHF, ABGK are described outwardly to the △. Prove that

$$DG^2 - EF^2 = 3\,(AB^2 - AC^2).$$

48. In the hypotenuse AB of a rt.-∠d △ ACB, points D and E are taken such that AD = AC and BE = BC. Prove that

$$DE^2 = 2\,BD\,.\,AE.$$

49. A st. line is 8 cm. in length. Divide it into two parts such that the difference of the squares on the parts = 5 sq. cm.

50. A and B are two given points and CD is a given st. line. Find a point P in CD such that the difference of the squares on PA and PB may be equal to a given rectangle.

51. AD is a median of the acute-∠d △ ABC; DX ⊥ AB, DY ⊥ AC. Prove that

$$BA\,.\,AX + CA\,.\,AY = 2\,AD^2.$$

52. Find a point P within a given quadrilateral KLMN such that △ PLM = △ PMN = △ PNK.

53. ABC is an isosceles △ in which AB = AC. AP ∥ BC. Prove that the difference between PB^2 and PC^2 equals 2 AP.BC.

54. If the sum of the squares on the diagonals of a quadrilateral be equal to the sum of the squares on the sides, the quadrilateral is a ||gm.

55. **D** is a point in the side **BC** of a $\triangle$ **ABC** such that $AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$. $AX \perp BC$. Prove that either $BD = DC$, or $2\,DX = BC$.

56. **ABCD** is a ||gm, and **P** is a point such that $PA^2 + PC^2 = PB^2 + PD^2$. Prove that **ABDC** is a rectangle.

57. **A, B, C, D** are four fixed points. Find the locus of a point **P** such that $PA^2 + PB^2 + PC^2 + PD^2$ is constant.

58. **A, B, C, D** are four fixed points. Find the locus of a point **P** such that $PA^2 + PB^2 = PC^2 + PD^2$.

59. **D** and **E** are taken in the base **BC** of $\triangle$ **ABC** so that $BD = EC$. Through **D, E** st. lines are drawn || **AB** and **AC** forming two ||gms with **AD, AE** as diagonals. Prove the ||gms equal in area.

60. A st. line **EF** drawn || to the diagonal **AC** of a ||gm **ABCD** meets **AB** in **E** and **BC** in **F**. Prove that **BD** bisects the quadrilateral **DEBF**.

61. **ABC** is an isosceles rt.-$\angle$d $\triangle$ in which $AB = AC$. **E** is taken in **AB** and **D** in **AC** produced such that $EB = CD$. Prove that $\triangle$ **EAD** $<$ $\triangle$ **ABC**.

62. **L** and **M** are respectively the middle points of the diagonals **BD** and **AC** of a quadrilateral **ABCD**. **ML** is produced to meet **AD** at **E**. Prove that $\triangle$ **EBC** = half the quadrilateral.

63. **DE** is || **BC** the base of $\triangle$ **ABC**, and meets **AB, AC** at **D, E** respectively. **DE** is produced to **F** making $DF = BC$. Prove that $\triangle$ **AEF** = $\triangle$ **BDE**.

64. Construct the minimum $\triangle$ which has a fixed vertical $\angle$, and its base passing through a fixed point situated between the arms of the $\angle$.

65. **BE, BD** are the bisectors of the interior and exterior $\angle$s at **B** in the $\triangle$ **ABC**. $AE \perp BE$ and $CD \perp BD$. **AE** and **CD** intersect at **F**. Prove that rect. **BEFD** = $\triangle$ **ABC**.

66. **ABCD** is a square. St. lines drawn through **A** and **D** make with **BC** produced in both directions the $\triangle$ **EFG**. **EX** $\perp$ **FG**. Prove that **BC** (**EX** + **FG**) = 2 $\triangle$ **EFG**.

67. The $\triangle$ **ABC** is rt.-$\angle$d at **C**, and the bisectors of $\angle$s **A** and **B** meet at **E**. **ED** $\perp$ **AB**. Prove that rect. **AD . DB** = $\triangle$ **ABC**.

68. Calculate the area of an equilateral $\triangle$ of which the side is 2 inches.

69. If the side of an equilateral $\triangle$ is a inches, show that its area is $\frac{a^2\sqrt{3}}{4}$ sq. in.

70. Calculate the side of an equilateral $\triangle$ of which the area is 10 sq. cm.

71. Construct a $\triangle$ having two sides 4 cm. and 4·5 cm., and the area 7 sq. cm.

Show that there are two solutions.

72. **M** is a point in the side **QR** of $\triangle$ **PQR** such that **QM** = 2 **MR**. Prove that $PQ^2 + 2\ PR^2 = 3\ PM^2 + 6\ MR^2$.

73. The rectangle contained by the two segments of a st. line is a maximum when the st. line is bisected. (Use Ex. 8 (1), § 81.)

74. The sum of the squares on the two segments of a st. line is a minimum when the st. line is bisected. (Use Ex. 8 (2), § 81.)

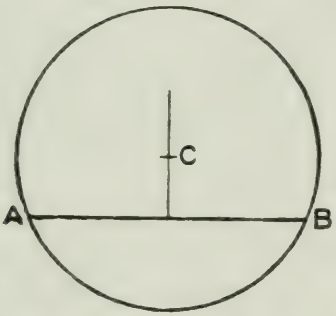
BOOK III

The Circle

89. A definition of a circle was given in § 32, and from the explanation given in § 66 we may take the following alternative definition of it:—

A **circle** is the locus of the points that lie at a fixed distance from a fixed point.

90. As the centre of a circle is a point equally distant from the two ends of any chord of the circle, the three following statements follow at once from I—22, p. 78:—



(*a*) The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.

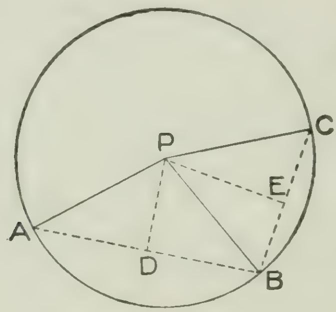
(*b*) The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.

(*c*) The right bisector of a chord of a circle passes through the centre of the circle.

As an exercise the pupil should give independent proofs of theorems (*a*), (*b*) and (*c*).

Theorem 1

If from a point within a circle more than two equal straight lines are drawn to the circumference, that point is the centre.



Hypothesis.—P is a point within the circle ABC such that PA = PB = PC.

To prove that P is the centre of the circle.

Construction. — Join AB, BC, and from P draw PD ⊥ AB and PE ⊥ BC.

Proof.—∵ PA = PB,

∴ P is in the right bisector of AB. (I—22, p. 78.)

And ∴ PD produced is the locus of the centres of all circles through A and B.

∴ the centre of the circle ABC is somewhere in PD.

In the same manner it may be shown that the centre of the circle ABC is somewhere in PE.

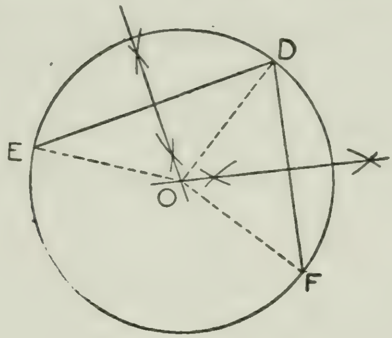
But P is the only point common to PD and PE.

∴ P is the centre of circle ABC.

Constructions

Problem 1

To find the centre of a given circle.



Let **DEF** be the given circle.

Construction.—From any point **D** on the circumference draw two chords **DE, DF**.

Draw the right bisectors of **DE, DF** meeting at **O**.

O is the centre of circle **DEF**.

Join **OF, OE, OF.**

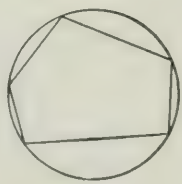
Proof.—$\because$ **O** is on the right bisector of **DE**,

$\therefore$ **OE = OD.** (I—22, p. 78.)

Similarly **OD = OF.**

$\because$ **OE = OD = OF,**

$\therefore$ **O** is the centre of the circle. (III—1, p. 142.)



91. **Definitions.**—If a circle passes through all the vertices of a rectilineal figure, it is said to be **circumscribed** about the figure.

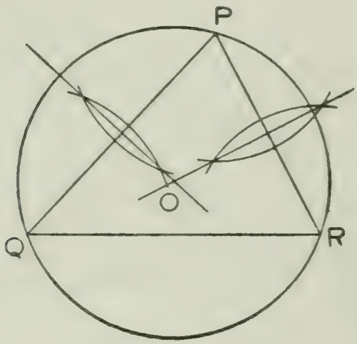
Four points so situated that a circle may be described to pass through all of them are said to be **concyclic.**

If the four vertices of a quadrilateral are on the circumference of the same circle, it is said to be a **cyclic quadrilateral.**

The centre of a circle circumscribed about a triangle is called the **circumcentre** of the triangle.

Problem 2

To circumscribe a circle about a given triangle.



Let PQR be the given △.

Construction.—Draw the right bisectors of PQ, PR meeting at O.

∵ O is on the right bisector of PQ.

∴ OP = OQ. (I—22, p. 78.)

Similarly OP = OR.

∴ OP = OQ = OR,

And a circle described with centre O and radius OP will pass through Q and R, and be circumscribed about the △.

92.—Exercises

1. Through a given point within a circle draw a chord that is bisected at the given point.

2. Complete a circle of which an arc only is given.

3. Circumscribe a circle about a given square.

4. Circumscribe a circle about a given rectangle.

5. Describe a circle with a given centre to cut a given circle at the ends of a diameter.

6. The locus of the middle points of a system of || chords in a circle is a diameter of the circle.

7. **If two circles cut each other, the st. line joining their centres bisects their common chord at rt. ∠s.**

8. If each of two equal st. lines has one extremity on one of two concentric circles, and the other extremity on the other circle, the st. lines subtend equal ∠s at the common centres.

9. A st. line cuts the outer of two concentric circles at E, F; and the inner at G, H. Prove that EG = FH.

10. **A st. line cannot cut a circle at more than two points.**

11. Two chords of a circle cannot bisect each other unless both are diameters.

12. A circle cannot be circumscribed about a ||gm unless the ||gm is a rectangle.

13. A st. line which joins the middle points of two || chords in a circle is ⊥ to the chords.

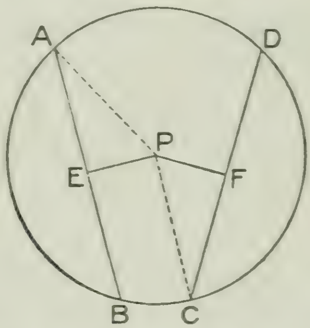
14. If two circles cut each other, a st. line through a point of intersection, || to the line of centres and terminated in the circumferences, is double the line joining the centres.

15. If two circles cut each other, any two $\parallel$ st. lines through the points of intersection, and terminated by the circumferences, are equal to each other.

16. If two circles cut each other, any two st. lines through one of the points of intersection, making equal $\angle$s with the line of centres and terminated by the circumferences, are equal to each other.

Theorem 2

Chords that are equally distant from the centre of a circle are equal to each other.



Hypothesis.—ABC is a circle of which P is the centre and AB, CD are two chords such that the $\perp$s PE, PF from P to AB, CD respectively are equal to each other.

To prove that AB = CD.

Construction.—Join AP, CP.

Proof.—Rotate $\triangle$ PFC about point P making PF fall on PE.

$\because$ PF = PE,

$\therefore$ point F falls on point E.

$\because$ $\angle$ PFC = $\angle$ PEA,

$\therefore$ FC falls along EA.

hence, $\because$ PC is a radius and

$\therefore$ C remains on the circumference,

C must fall on A.

$\therefore$ FC coincides with EA,

and $\therefore$ $FC = EA$,

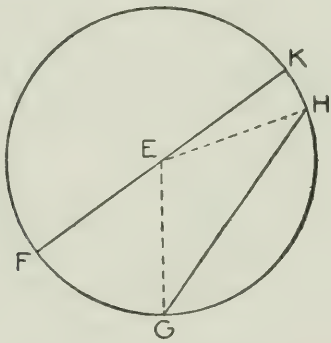
But $CD = 2\ CF$,

and $AB = 2\ AE$,

$\therefore$ $CD = AB$.

THEOREM 3

In a circle any chord which does not pass through the centre is less than a diameter.



Hypothesis.—In the circle FGH, GH is a chord which does not pass through the centre and FK is a diameter. E is the centre.

To prove that $GH < FK$.

Construction.—Join EG, EH.

Proof.— $\because$ $GE = EF$ and $EH = EK$,

$\therefore$ $GE + EH = FK$.

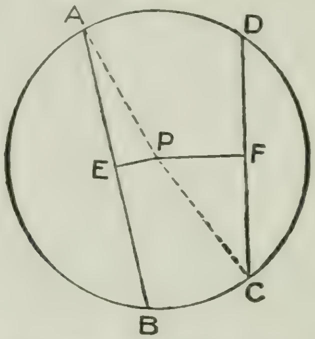
$\because$ GEH is a $\triangle$,

$\therefore$ $GH < GE + EH$. (I—16, p. 59.)

And $\therefore$ $GH < FK$.

Theorem 4

Of two chords in a circle the one which is nearer to the centre is greater than the one which is more remote from the centre.



Hypothesis.—P is the centre of a circle ABC, and AB, CD are two chords such that PE, the distance of AB from the centre, is less than PF, the distance of CD from the centre.

To prove that AB $>$ CD.

Construction.—Join PA, PC.

Proof.— $\because$ PEA is a rt. $\angle$,

$\therefore$ $AE^2 + EP^2 = AP^2$. (II—13, p. 122.)

Similarly $CF^2 + FP^2 = CP^2$.

But $\because$ $AP = CP$,

$\therefore$ $AP^2 = CP^2$.

And $\therefore$ $AE^2 + EP^2 = CF^2 + FP^2$.

$\because$ $EP < PF$,

$\therefore$ $EP^2 < PF^2$.

And $\therefore$ $AE^2 > CF^2$,

$\therefore$ $AE > CF$.

But $AB = 2\,AE$,

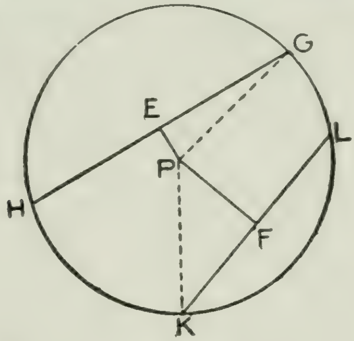
and $CD = 2\,CF$,

$\therefore$ $AB > CD$.

Theorem 5

(Converse of Theorem 4)

If two chords of a circle are unequal, the greater is nearer to the centre than the less.



Hypothesis.—Chord GH $>$ chord KL, and PE, PF are respectively perpendiculars from the centre P to GH, KL.

To prove that PE $<$ PF.

Construction.—Join PG, PK.

Proof.— $\because$ PEG is a rt. $\angle$,

$GE^2 + EP^2 = GP^2$. (II—13, p. 122.)

Similarly $PF^2 + FK^2 = PK^2$.

$\therefore GE^2 + EP^2 = PF^2 + FK^2$.

But $\because GH = 2\,GE$ and $KL = 2\,KF$,

And also $GH > KL$,

$\therefore GE > KF$.

$\therefore GE^2 > KF^2$.

Hence, $EP^2 < PF^2$.

And $\therefore EP < PF$.

93.—Exercises

1. **If two chords of a circle are equal to each other, they are equally distant from the centre.** (Converse of Theorem 2.)

2. A chord 6 cm. in length is placed in a circle of radius 4 cm. Calculate the distance of the chord from the centre.

3. A chord a inches long is placed in a circle of radius b inches. Find an algebraic expression for the distance of the chord from the centre.

4. In a circle of radius 5 cm. a chord is placed at a distance of 3 cm. from the centre. Calculate the length of the chord.

5. Through a given point within a circle draw the shortest chord.

6. In a circle of radius 4 cm., a point **P** is taken at the distance 3 cm. from the centre. Calculate the length of the shortest chord through **P**.

7. The length of a chord 2 cm. from the centre of a circle is 5·5 cm. Find the length of a chord 3 cm. from the centre. Verify your result by measurement.

8. In a circle of radius 5 cm., two || chords of lengths 8 cm. and 6 cm. are placed. Find the distance between the chords. Show that there are two solutions.

9. **ACB** is a diameter, and **C** the centre of a circle. **D** is any point on **AB**, or on **AB** produced, and **P** is any point on the circumference except **A** and **B**. Show that **DP** is intermediate in magnitude between **DA** and **DB**.

10. **O** is the centre of a circle, and **P** is any point. If two st. lines be drawn through **P**, cutting the circle, and

making equal $\angle$s with PO, the chords intercepted on these lines by the circumference are equal to each other.

11. O is the centre of a circle, and P is any point. On two lines drawn through P chords AB, CD are intercepted by the circumference. If the $\angle$ made by AB with PO $>$ $\angle$ made by CD with PO, the chord AB $<$ chord CD.

12. From any point in a circle which is not the centre equal st. lines can be drawn to the circumference only in pairs.

13. Find the locus of the middle points of chords of a fixed length in a circle.

14 K and L are two fixed points. Find a point P on a given circle such that $KP^2 + LP^2$ may be the least possible.

15. Chords equally distant from the centre of a circle subtend equal $\angle$s at the centre.

16. The nearer to the centre of two chords of a circle subtends the greater $\angle$ at the centre.

ANSWERS :—2, 26·5 mm. nearly ; 4, 8 cm.; 6, 5·3 c.m. nearly ; 7, 32 mm. nearly ; 8, 1 cm. or 7 cm.

Angles in a Circle

Theorem 6

The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.

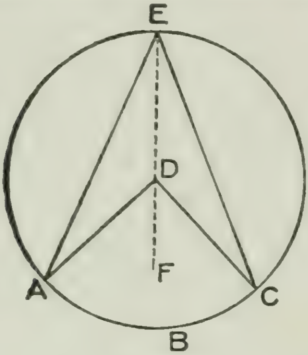


Fig. 1

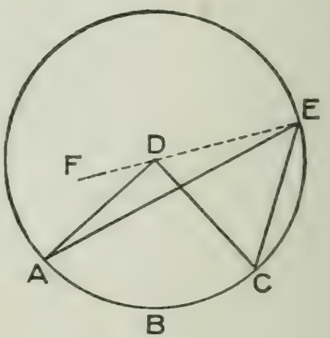


Fig. 2

Hypothesis.—ABC is an arc of a circle, D the centre, and E any point on the remaining part of the circumference.

To prove that $\angle$ ADC = 2 $\angle$ AEC.

Construction. — Join ED and produce ED to any point F.

Proof.—

In both figures:—

In $\triangle$ DAE, $\because$ DA = DE

$\therefore$ $\angle$ DAE = $\angle$ DEA (I—3, p. 20).

$\because$ ADF is an exterior $\angle$ of $\triangle$ ADE,

$\therefore$ $\angle$ ADF = $\angle$ DAE + $\angle$ DEA (I—10, p. 45.)

= 2 $\angle$ DEA.

Similarly $\angle$ CDF = 2 $\angle$ DEC.

In Fig. 1:—

$$\angle ADF = 2 \angle DEA$$
$$\angle CDF = 2 \angle DEC,$$
adding, $\angle ADC = 2(\angle DEA + \angle DEC)$
$$= 2 \angle AEC.$$

In Fig. 2:—

$$\angle CDF = 2 \angle DEC,$$
$$\angle ADF = 2 \angle DEA,$$
subtracting, $\angle ADC = 2(\angle DEC - \angle DEA).$
$$= 2 \angle AEC.$$

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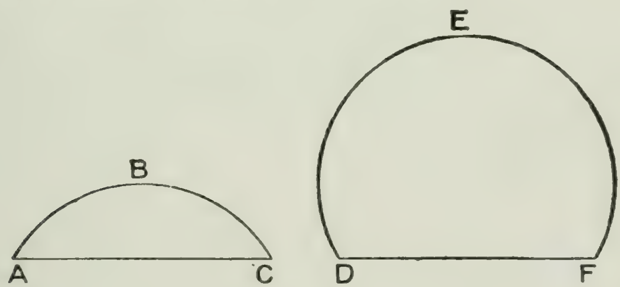
94. **Definitions.**—The figure bounded by an arc of a circle and the chord which joins the ends of the arc is called a **segment of a circle.**



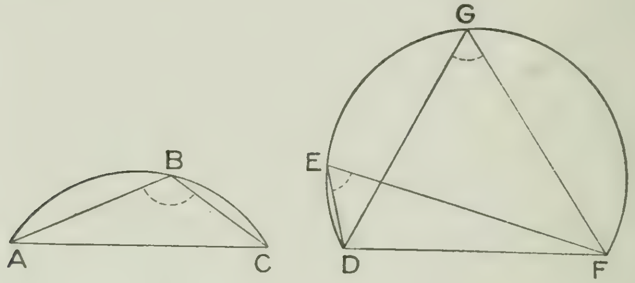
ABC, DEF are segments of circles.

A semi-circle is a particular case of a segment.

An arc is called a **major arc** or a **minor arc** according as it is greater or less than half the circumference.

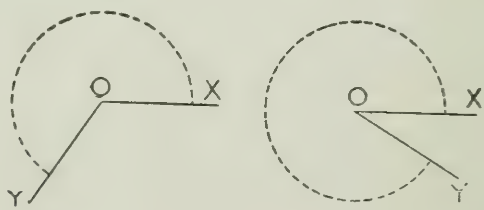
A segment is called a **major segment** or a **minor segment** according as the arc of the segment is a major or a minor arc.

95. **Definitions.**—If the ends of a chord of a segment are joined to any point on the arc of the segment, the angle between the joining lines is called an **angle in the segment.**



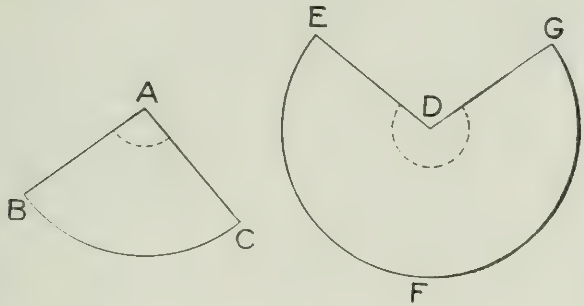
ABC is an ∠ **in** the segment **ABC,** and **DEF** is an ∠ **in** the segment **DEF.** **DGF** is also an ∠ **in** the segment **DEF.**

96. **Definitions.**—An angle which is greater than two right angles but less than four right angles is called a **reflex** angle.



A straight line starting from the position **OX** and rotating in the direction opposite to that of the hands of a clock to the position **OY,** in either diagram, traces out the reflex angle **XOY.**

The figure bounded by two radii of a circle, and either of the arcs intercepted by the radii is called a **sector of the circle.**

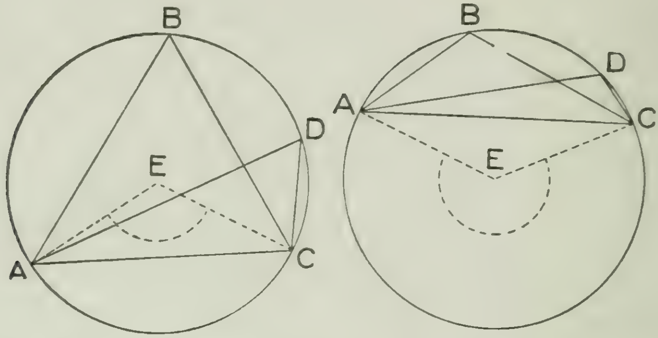


ABC, DEFG are sectors of circles.

BAC is the ∠ of the sector ABC, and the reflex ∠ EDG is the ∠ of the sector DEFG.

Theorem 7

Angles in the same segment of a circle are equal to each other.



Hypothesis.—ABC, ADC are two ∠s in the same segment ABDC.

To prove that ∠ ABC = ∠ ADC.

Construction.—Find E the centre of the circle. Join AE, EC.

Proof.—The ∠ AEC at the centre and the ∠s ABC and ADC at the circumference are subtended by the same arc,

$\therefore$ ∠ ABC = $\frac{1}{2}$ ∠ AEC, (III—6, p. 152.)

and ∠ ADC = $\frac{1}{2}$ ∠ AEC,

$\therefore$ ∠ ABC = ∠ ADC.

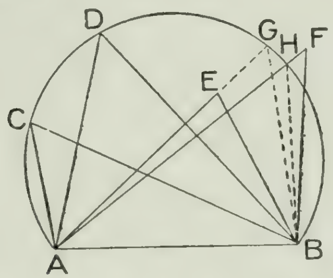
Alternative statement of the preceding theorem:—

The angle in a given segment is constant in magnitude for all positions of the vertex of the angle on the arc of the segment.

THEOREM 8

(Converse of Theorem 7)

The locus of all points on one side of a straight line at which the straight line subtends equal angles is the arc of a segment of which the straight line is the chord.



Hypothesis.—AB is a st. line, and C one of the points. Circumscribe a circle about the $\triangle$ ACB.

To prove that arc ACB is the locus of all points on the same side of AB at which AB subtends $\angle$s equal to $\angle$ ACB.

Construction.—Take any other point D on arc ACB, E any point within the segment, and F any point without the segment.

Join AD, DB, AE, EB, AF, FB.

Proof.—Then $\angle$ ADB = $\angle$ ACB. (III—7, p. 156.)

Produce AE to meet arc ACB at G. Join BG.

$\because$ AEB is an exterior $\angle$ of $\triangle$ EGB,

$\therefore$ $\angle$ AEB $>$ $\angle$ AGB; (I—10, Cor., p. 45.)

but $\angle$ AGB = $\angle$ ACB, (III—7, p. 156.)

$\therefore$ $\angle$ AEB $>$ $\angle$ ACB;

In a similar manner it may be shown that

$\angle$ AFB $<$ $\angle$ ACB;

and consequently arc ACB is the locus.

97. **Definition:**—If the three angles of one triangle are respectively equal to the three angles of another triangle, the triangles are said to be **similar.**

98. There are two conditions implied when figures are said to be similar: not only are the angles of one respectively equal to the angles of the other, but a certain relationship must exist between the lengths of the sides of the two figures. For triangles, it will be shown in Book IV that, if one of these conditions is given, the other is also true. For figures of more than three sides this is not the case, and a definition including both conditions must be given. (See § 131.)

The symbol ||| may be used for the word similar, or for "is similar to."

99.—Exercises

1. Prove Theorem 6 when the arc is half the circumference.

2. Construct a circular arc on a chord of 3 inches and having the apex 3 inches from the chord. Calculate the radius of the circle.

3. If the chord of an arc is a inches, and the distance of its apex from the chord b inches, show that the radius of the circle is $\frac{a^2 + 4b^2}{8b}$.

4. Two chords **AOB, COD,** intersect at a point **O** within the circle. Show that **AOC, BOD** are similar $\triangle$s. **BOC, AOD** are also similar $\triangle$s. Read the segments that contain the equal $\angle$s.

5. **ABC** is a $\triangle$ inscribed in a circle, and the bisector of $\angle$ **A** meets the circumference again at **D.** Show that the st. line drawn from **D** $\perp$ **BC** is a diameter.

6. A circle is divided into two segments by a chord equal to the radius. Show that the ∠ in the major segment is 30° and that in the minor segment is 150°.

7. The locus of the vertices of the rt. ∠s of all rt.-∠d △s on the same hypotenuse is a circle.

8. Prove Theorem 6 when the arc is greater than half the circumference.

9. **PQR** is a △ inscribed in a circle. The bisector of ∠ **P** cuts **QR** at **D** and meets the circle at **E**. Prove that △ **PQD** ||| △ **PER**.

10. **DPQ** and **EPQ** are two fixed circles, and **D**, **P** and **E** are in the same st. line. The bisector of ∠ **DQE** meets **DE** at **F**. Show that the locus of **F** is an arc of a circle.

11. If the diagonals of a quadrilateral inscribed in a circle cut at rt. ∠s, the ⊥ from their intersection on any side bisects the opposite side.

12. If the diagonals of a quadrilateral inscribed in a circle cut at rt. ∠s, the distance of the centre of the circle from any side is half the opposite side.

13. If the diagonals of a quadrilateral inscribed in a circle cut each other at rt. ∠s, the ∠s which a pair of opposite sides of the quadrilateral subtend at the centre of the circle are supplementary.

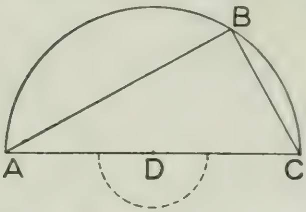
14. **XYZ**, **XYV** are two equal circles, the centre of each being on the circumference of the other. **ZXV** is a st. line. Prove that **YZV** is an equilateral △.

15. **EFGH** is a quadrilateral inscribed in a circle and **EF** = **GH**. Prove that **EG** = **FH**.

16. **ABCD** is a quadrilateral inscribed in a circle; the diagonals **AC**, **BD** cut at **E**; **F** the centre of the circle is within the quadrilateral. Prove that ∠ **AFB** + ∠ **CFD** = 2 ∠ **AEB**.

Theorem 9

The angle in a semi-circle is a right angle.



Hypothesis.—**ABC** is an $\angle$ in the semi-circle **ABC**, of which **D** is the centre.

To prove that **ABC** is a rt. $\angle$.

Proof.—The $\angle$ **ABC** at the circumference, and the st. $\angle$ **ADC** at the centre, would each subtend the same arc, if the circle were complete.

$\therefore \angle$ **ABC** $= \frac{1}{2} \angle$ **ADC**. (III—6, p. 152.)

$=$ a rt. $\angle$.

THEOREM 10

(a) The angle in a major segment of a circle is acute.

(b) The angle in a minor segment of a circle is obtuse.

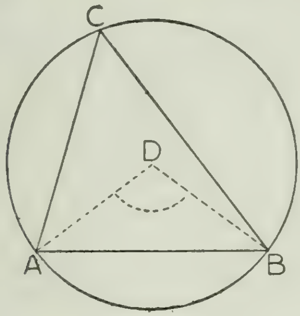


Fig 1

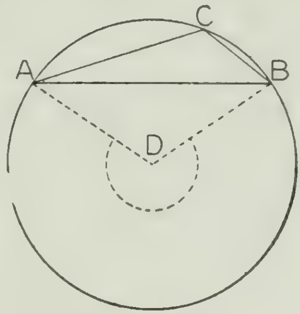


Fig.2

(a) *Hypothesis.*—ACB is an ∠ in a major segment of a circle. (Fig. 1.)

To prove that ∠ ACB is acute.

Construction.—Join A and B to the centre D.

Proof.—∠ ACB at the circumference and ∠ ADB at the centre stand on the same arc,

∴ ∠ ACB = $\frac{1}{2}$ ∠ ADB. (III—6, p. 152.)

But ∠ ADB is < a st. ∠.

∴ ∠ ACB is acute.

(b) *Hypothesis.*—ACB is an ∠ in a minor segment of a circle. (Fig. 2.)

To prove that ∠ ACB is obtuse.

Construction.—Join A and B to the centre D.

Proof.—

∠ ACB = $\frac{1}{2}$ the reflex ∠ ADB. (III—6, p. 152.)

∴ ∠ ACB is obtuse.

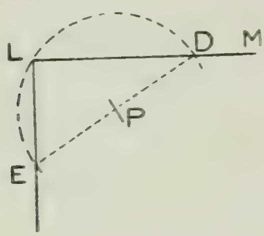
100.—Exercises

1. **A circle described on the hypotenuse of a rt.-∠d △ as diameter passes through the vertex of the rt. ∠.** (Converse of III – 9).

2. Circles described on two sides of a △ as diameters, intersect on the third side, or the third side produced.

Where is the point of intersection when the circles are described on the equal sides of an isosceles △?

3. LM is a st. line and L a point from which it is required to draw a ⊥ to LM.



Construction.—With a convenient point P as centre describe a circle to pass through L and cut LM at D. Join DP, and produce DP to cut the circle at E. Join LE.

Prove LE ⊥ LM.

4. EF, EG are diameters of two circles FEH, GEH respectively. Show that FHG is a st. line.

5. ST is a diameter of the circle SVT. A circle is described with centre S and radius ST. Show that any chord of this latter circle drawn from T is bisected by the circle SVT.

6. Chords of a given circle are drawn through a given point. Find the locus of the middle points of the chords when the given point is (a) on the circumference, (b) within the circle, (c) without the circle.

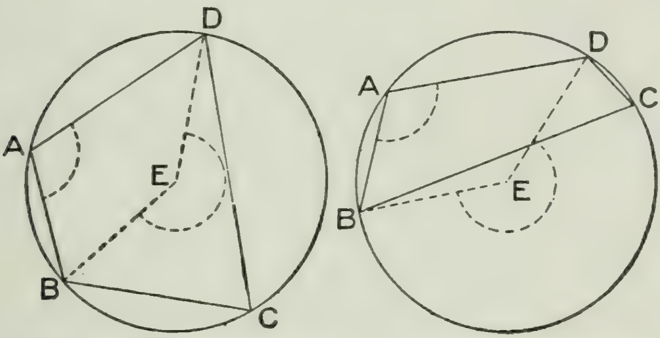
7. F is any point on the arc of a semi-circle of which DE is a diameter. The bisectors of ∠s FED, FDE meet at P. Find the locus of P.

8. **F** is a point on the arc of a semi-circle of which **DE** is a diameter. **FG** $\perp$ **DE**. Show that the $\triangle$s **FDG**, **FEG**, **FDE** are similar.

9. **PQRS** is a st. line and circles described on **PR**, **QS** as diameters cut at **E**. Prove that $\angle$ **PEQ** = $\angle$ **RES**.

Theorem 11

If a quadrilateral is inscribed in a circle, its opposite angles are supplementary.



Hypothesis.—**ABCD** is a quadrilateral inscribed in a circle.

To prove that $\angle$ **A** + $\angle$ **C** = 2 rt. $\angle$s.

Construction.—Find the centre **E**. Join **BE**, **ED**.

Proof.—$\angle$ **BED** at the centre and $\angle$ **C** at the circumference are subtended by the same arc **BAD**.

$\therefore$ $\angle$ **C** = $\frac{1}{2}$ $\angle$ **BED**. (III—6, p. 152.)

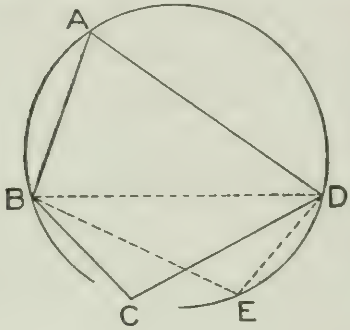
Similarly $\angle$ **A** = $\frac{1}{2}$ reflex $\angle$ **BED**.

Hence $\angle$ **A** + $\angle$ **C** = $\frac{1}{2}$ the sum of the two $\angle$s **BED** at the centre = $\frac{1}{2}$ of 4 rt. $\angle$s

= 2 rt. $\angle$s.

Theorem 12

If the opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Hypothesis.—**ABCD** is a quadrilateral in which $\angle$ **A** + $\angle$ **C** = 2 rt. $\angle$s.

To prove that **A, B, C, D** are on the circumference of a circle.

Construction.—Draw a circle through the three points **A, B, D**. On this circumference and on the side of **BD** remote from **A** take a point **E**. Join **BE, ED**.

Proof.—$\because$ **ABED** is a quadrilateral inscribed in a circle,

$\therefore$ $\angle$ **A** + $\angle$ **E** = 2 rt. $\angle$s; (III—11, p. 163.)

but $\angle$ **A** + $\angle$ **C** = 2 rt. $\angle$s. (*Hyp.*)

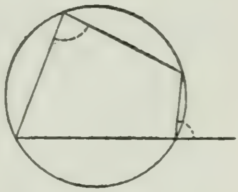
$\therefore$ $\angle$ **A** + $\angle$ **E** = $\angle$ **A** + $\angle$**C**,

and $\therefore$ $\angle$ **E** = $\angle$ **C**.

Consequently, as **C, E** are on the same side of **BD**, the circle **BADE** passes through **C**. (III—8, p. 157.)

101.—Exercises

1. **If one side of an inscribed quadrilateral be produced, the exterior ∠ thus formed at one vertex equals the interior ∠ at the opposite vertex of the quadrilateral.**



State and prove the converse.

2. From a point **O** without a circle two st. lines **OAB**, **OCD** are drawn cutting the circumference at **A, B, C, D**. Show that △s **OBC**, **OAD** are similar, and that △s **OAC**, **OBD** are similar.

3. If a ||gm be inscribed in a circle, the ||gm is a rect.

4. **A, D, C, E, B** are five successive points on the circumference of a circle; and **A, B** are fixed. Show that the sum of the ∠s **ADC**, **CEB** is the same for all positions of **D, C, E**.

5. A circle is circumscribed about an equilateral △. Show that the ∠ in each segment outside the △ is an ∠ of 120°.

6. A scalene △ is inscribed in a circle. Show that the sum of the ∠s in the three segments outside the △ is 360°.

7. A quadrilateral is inscribed in a circle. Show that the sum of the ∠s in the four segments outside the quadrilateral is 540°.

8. **P** is a point on the diagonal **KM** of the ||gm **KLMN**. Circles are described about **PKN** and **PLM**. Show that **LN** passes through the other point of intersection of the circles.

9. A circle drawn through the middle points of the sides of a △ passes through the feet of the ⊥s from the vertices to the opposite sides.

10. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet at **L** and **M**, and about the △s

so formed outside the quadrilateral circles be described intersecting again at N, then L, M, N are in the same st. line.

11. In a $\triangle$ DEF, DX $\perp$ EF and EY $\perp$ DF. Prove that $\angle$ XYF = $\angle$ DEF.

12. PQRS, PQTV are circles and SPV, RQT are st. lines. Prove that SR $\parallel$ VT.

13. The st. lines that bisect any $\angle$ of a quadrilateral inscribed in a circle and the opposite exterior $\angle$ meet on the circumference.

14. XYZ is a $\triangle$; YD $\perp$ ZX, and DE $\perp$ XY; ZF $\perp$ XY and FG $\perp$ ZX. Show that EG $\parallel$ YZ

15. EGD, FGD are two circles with centres H, K respectively. EGF is a st. line. EH, FK meet at P. Show that H, K, D, P are concyclic.

16. KL, MN are two $\parallel$ chords in a circle; KE, NF two $\perp$ chords in the same circle. Show that LF $\perp$ ME.

17. The bisectors of the $\angle$s formed by producing the opposite sides of a quadrilateral inscribed in a circle are $\perp$ to each other.

18. HKM, LKM are two circles, and HKL is a st. line. HM, LM cut the circles again at E, F respectively, and HF cuts LE at G. Show that a circle may be circumscribed about MEGF.

19. PQRS is a quadrilateral and the bisectors of the $\angle$s P, Q; Q, R; R, S; S, P meet at four points. Show that a circle may be circumscribed about the quadrilateral thus formed.

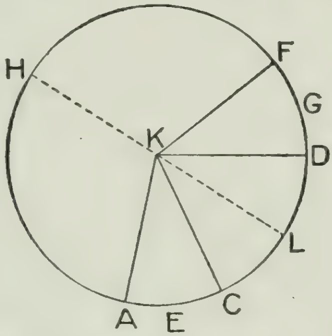
20. EF is the diameter of a semi-circle and G, H any two points on its arc. EH, FG cut at K and EG, FH cut at L. Show that KL $\perp$ EF.

21. **DE** is the diameter, **O** the centre and **P** any point on the arc of a semi-circle. **PM** $\perp$ **DE.** Show that the bisector of $\angle$ **MPO** passes through a fixed point.

22. **PQR** is a $\triangle$ and **PDQ, PFQ** are two circles cutting **PR** at **D, F** and **QR** at **E, G** Prove that **DE** $\parallel$ **FG.**

Theorem 13

If two angles at the centre of a circle are equal to each other, they are subtended by equal arcs.



Hypothesis.—**AKC, DKF** are equal $\angle$s at the centre **K** of the circle **ACD.**

To prove that arc **AEC** equals arc **DGF.**

Construction. — Draw the diameter **HKL** bisecting $\angle$ **CKD.**

Proof.—Suppose the circle to be folded along the diameter **HKL**, and the semi-circle **HFL** will coincide throughout with the semi-circle **HAL.**

$\because$ $\angle$ **LKD** = $\angle$ **LKC,**

$\therefore$ **KD** falls along **KC**;

and $\therefore$ **D** falls on **C.**

$\because$ $\angle$ **DKF** = $\angle$ **CKA,**

$\therefore$ **KF** falls along **KA**;

and $\therefore$ **F** falls on **A.**

$\therefore$ the arc **DGF** coincides with the arc **CEA.**

$\therefore$ arc **DGF** = arc **CEA.**

102.– Exercises

1. **If two arcs of a circle be equal to each other, they subtend equal ∠s at the centre.** (Prove either by indirect demonstration, or by the construction and method used in III—13.)

2. **If two ∠s at the circumference of a circle be equal to each other, they are subtended by equal arcs.**

3. **If two arcs of a circle be equal to each other, they subtend equal ∠s at the circumference.**

4. In equal circles equal ∠s at the centres (or circumferences) stand on equal arcs.

5. In equal circles equal arcs subtend equal ∠s at the centres (or circumferences).

6. **If two arcs of a circle (or of equal circles) be equal, they are cut off by equal chords.**

7. **If two chords of a circle be equal to each other, the major and minor arcs cut off by one are respectively equal to the major and minor arcs cut off by the other.**

8. If two sectors of a circle have equal ∠s at the centre, the sectors are congruent.

9. **Bisect a given arc of a circle.**

10. Parallel chords of a circle intercept equal arcs.

Show also that the converse is true.

11. If two equal circles cut one another, any st. line drawn through one of the points of intersection will meet the circles again at two points which are equally distant from the other point of intersection.

12. The bisectors of the opposite ∠s of a quadrilateral inscribed in a circle meet the circumference at the ends of a diameter.

13. If two ∠s at the centre of a circle be supplementary, the sum of the arcs on which they stand is equal to half the circumference.

14. If any number of ∠s be in a segment, their bisectors all pass through one point.

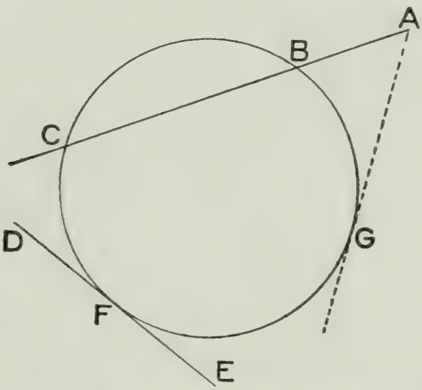
Tangents and Chords

103. **Definitions.**—Any straight line which cuts a circle is called a **secant.**

A straight line which, however far it may be produced, has one point on the circumference of a circle, and all other points without the circle is called a **tangent** to the circle.

A tangent is said to **touch** the circle.

The common point of a tangent and circle, that is, the point where the tangent touches the circle, is called the **point of contact.**



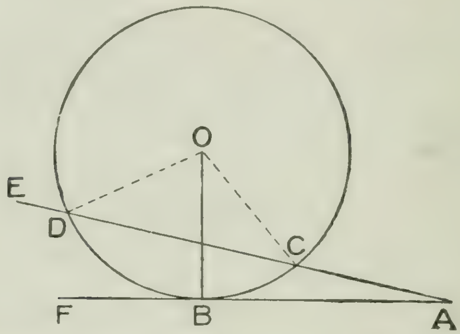
ABC is a secant drawn to the circle BCF from the point A.

DFE is a tangent to the circle BCF, touching the circle at the point of contact F.

If the secant ABC rotate about the point A until the two points B, C where it cuts the circle coincide at G, the secant becomes a tangent having G for the point of contact.

Theorem 14

The radius drawn to the point of contact of a tangent is perpendicular to the tangent.



Hypothesis.—**ABF** is a tangent to the circle **CBD** at the point **B**, **O** is the centre and **OB** the radius drawn to the point of contact.

To prove that **OB** is $\perp$ **AF**.

Construction. — From any point **A**, except **B**, in **AF** draw a secant **AE** cutting the circle in **C** and **D**. Join **OC**, **OD**.

Proof.— $\because$ $OD = OC$,

$\therefore \angle ODC = \angle OCD$. (I—3, p. 20.)

But, st. $\angle EDC =$ st. $\angle DCA$,

$\therefore \angle ODE = \angle OCA$.

Rotate **AE** about **A** until it coincides with **AF**. As **AE** rotates about **A** the $\angle$s **ODE**, **OCA** are continually equal to each other and finally $\angle$ **ODE** becomes $\angle$ **OBF** and $\angle$ **OCA** becomes $\angle$ **OBA**.

$\therefore \angle OBF = \angle OBA$.

and $\therefore$ $OB \perp AF$.

Cor. 1.—Only one tangent can be drawn at any point on the circumference of a circle.

∵ only one st. line can be ⊥ to the radius at that point.

Hence, also:—**The straight line drawn perpendicular to a radius at the point where it meets the circumference is a tangent.**

Cor. 2.—The perpendicular to a tangent at its point of contact passes through the centre of the circle.

∴ only one st. line can be ⊥ to the tangent at that point.

Cor. 3.—The perpendicular from the centre on a tangent passes through the point of contact.

∵ only one ⊥ can be drawn from a given external point to a given st. line.

104.—Exercises

1. Draw a tangent to a given circle from a given point on the circumference.

2. Describe a circle with its centre on a given st. line **DE** to pass through a given point **P** in **DE** and touch another given st. line **DF**.

3. Find the locus of the centres of all circles that touch a given st. line at a given point.

4. Describe a circle to pass through a given point and touch a given st. line at a given point.

5. Tangents at the ends of a diameter are ||.

6. **C** is any point on the tangent of which **A** is the point of contact. The st. line from **C** to the centre **O** cuts the circumference at **B**. **AD** is ⊥ **OC**. Show that **BA** bisects the ∠ **DAC**.

7. Find the locus of the centres of all circles which touch two given || st. lines.

8. Draw a circle to touch two given || st. lines and pass through a given point between the ||s. Show that two such circles may be drawn.

9. To a given circle draw two tangents, each of which is || to a given st. line.

10. To a given circle draw two tangents, each of which is ⊥ to a given st. line.

11. Give an alternative proof for III—14 by supposing the radius **OB** drawn to the point of contact of the tangent **ABF** not ⊥ to **AF** and drawing **OG** ⊥ **AF**.

12. **Two tangents to a circle meet each other. Prove that they are equal to each other.**

13. **EF** is a diameter of a circle and **EG** is a chord. **EH** is a chord bisecting the ∠ **FEG**. Prove that the tangent at **H** is ⊥ **EG**.

14. Draw a circle to touch a given st. line at a given point and have its centre on another given st. line.

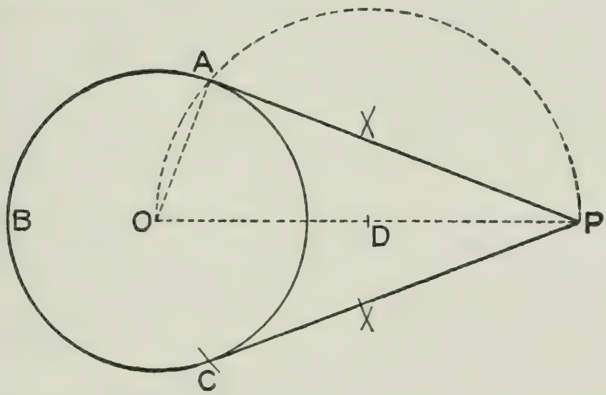
15. Draw a tangent to a given circle making a given ∠ with a given st. line.

Show that, in general, four such tangents may be drawn.

Construction

Problem 3

To draw a tangent to a given circle from a given point without the circle.



Let ABC be the given circle, and P the given point.

It is required to draw a tangent from P to the circle ABC.

Join P to the centre O. Bisect OP at D. With centre D and radius DO, describe a circle cutting the circle ABC at A and C. Join PA, PC.

Either PA or PC is a tangent to the given circle.

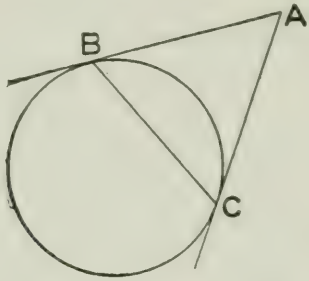
Join OA.

OAP is an ∠ in a semi-circle, and is ∴ a rt. ∠. (III—9, p. 160.)

∴ PA is a tangent. (III—14, Cor. 1, p. 171.)

In the same manner it may be shown that PC is a tangent.

105. **Definition.**—The straight line joining the points of contact of two tangents to a circle is called the **chord of contact** of the tangents.



BC is the chord of contact of the tangent **AB**, **AC**.

106.—Exercises

1. Draw a circle of radius 4 cm. Take a point 9 cm. from the centre of the circle. From this point draw two tangents to the circle. Measure the length of each tangent and check your result by calculation.

2. Draw a circle of radius 5 cm. Mark a point 7 cm. from the centre. From this point draw two tangents to the circle and measure the $\angle$ between the tangents. (91° nearly.)

3. Draw a circle with a radius of 3 cm. Mark any point **A** on the circumference, and from this point draw a tangent **AB** 4 cm. long. Measure the distance of **B** from the centre and check your result.

4. Draw a circle with 43 mm. radius. Draw any st. line through the centre, and find a point, in this line, from which the tangent to the circle will be 5 cm. in

length. Measure the distance of the point from the centre and check your result.

5. Mark two points **A** and **B** 7 cm. apart. Draw two st. lines from **A** such that the length of the perpendicular from **B** to either of them is 4 cm.

6. Draw a circle of radius 6 cm. Mark a point **P** 4 cm. from the centre. Draw a chord through **P** such that the perpendicular from the centre to the chord is 3 cm. in length. Measure the length of the chord and check your result by calculation.

7. Draw a circle of radius 36 mm. Mark any point **P** without the circle. Draw a st. line from **P** such that the chord cut off on it by the circle is 4 cm. in length.

8. Draw a circle of radius 47 mm. Mark a point **P** 4 cm. from the centre. Draw two chords through **P**, each of which is 65 mm. in length.

9. **If from a point without a circle two tangents be drawn, the st. line drawn from this point to the centre bisects the chord of contact and cuts it at rt. ∠s.**

10. If a quadrilateral be circumscribed about a circle, the sum of one pair of opposite sides equals the sum of the other pair.

11. Through a given point draw a st. line, such that the chord intercepted on the line by a given circle is equal to a given st. line.

12. If a ∥gm be circumscribed about a circle, the ∥gm is a rhombus.

13. If two tangents to a circle be ∥, their chord of contact is a diameter.

14. If two || tangents to a circle be cut by a third tangent to the circle at A, B; show that AB subtends a rt. ∠ at the centre.

15. If a quadrilateral be circumscribed about a circle, the ∠s subtended at the centre by a pair of opposite sides are supplementary.

16. To a given circle draw two tangents containing an ∠ equal to a given ∠.

17. Find the locus of the points from which tangents drawn to a given circle are equal to a given st. line.

18. Find a point P in a given st. line, such that the tangent from P to a given circle is of given length. What is the condition that this is possible?

19. E is a point outside a circle the centre of which is D. In DE produced find a point F, such that the length of the tangent from F may be twice that of the tangent from E.

20. Two tangents, LM, LN are drawn to a circle; P is any point on the circumference outside the △ LMN. Prove that ∠ LMP + ∠ LNP is constant.

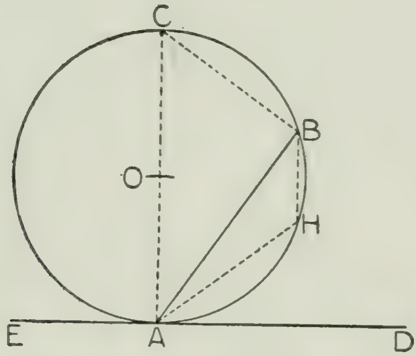
21. Find the ∠ between the tangents to a circle from a point whose distance from the centre is equal to a diameter.

22. Show that all equal chords of a given circle touch a fixed concentric circle.

23. From a given point without a circle draw a st. line such that the part intercepted by the circle subtends a rt. ∠ at the centre.

THEOREM 15

If at one end of a chord of a circle a tangent is drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Hypothesis.—AB is a chord and EAD a tangent to the circle ABC.

To prove that ∠ DAB = ∠ ACB and that ∠ EAB = ∠ AHB.

Construction.—From A draw the diameter AOC. Join BC. Join any point H in the arc AHB to A and B.

Proof.—∵ ABC is an ∠ in a semi-circle,

∴ ABC is a rt. ∠. (III—9, p. 160.)

∴ ∠ BAC + ∠ BCA = a rt. ∠ (I—10, p. 45.)

= ∠ CAD. (III—14, p. 170.)

Take away the common ∠ BAC,

∴ ∠ BAD = ∠ ACB,

= ∠ in the segment ACB.

∵ AHBC is an inscribed quadrilateral,

∴ ∠ H + ∠ C = a st. ∠ (III—11, p. 163.)

= st. ∠ DAE.

But ∠ C = ∠ BAD.

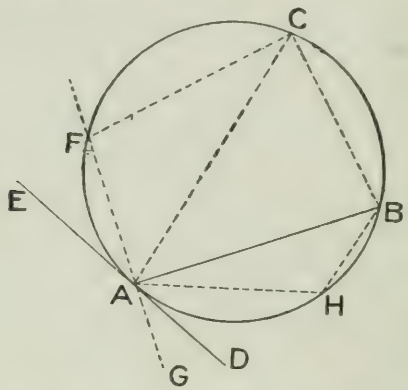
∵ ∠ BAE = ∠ H

= ∠ in the segment AHB.

Theorem 15

(Alternative Proof)

If at one end of a chord of a circle a tangent is drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Hypothesis.—AB is a chord and EAD a tangent to the circle ABC.

To prove that $\angle$ DAB = $\angle$ ACB, and that $\angle$ EAB = $\angle$ AHB.

Construction.—In arc AFC take any point F. Join CF, and draw the line FAG.

Proof.—$\because$ AFCB is an inscribed quadrilateral,

$\therefore$ $\angle$ FCB is supplementary to $\angle$ FAB,

(III—11, p. 163.)

But, $\angle$ BAG is supplementary to $\angle$ FAB.

$\therefore$ $\angle$ BAG = $\angle$ FCB.

These $\angle$s are equal however near F is to A.

Let F move along the circumference towards A and finally coincide with A.

The line **FAG** rotates about the point **A** and finally coincides with **EAD**. The $\angle$ **GAB** becomes $\angle$ **DAB** and $\angle$ **FCB** becomes $\angle$ **ACB**.

$\therefore$ $\angle$ **DAB** = $\angle$ **ACB**.

$\because$ $\angle$ **EAB** is supplementary to $\angle$ **DAB**,

and, $\angle$ **AHB** is supplementary to $\angle$ **ACB**.

(III—11, p. 163.)

$\therefore$ $\angle$ **EAB** = $\angle$ **AHB**.

107.—Exercises

1. **AB** is a chord of a circle and **AC** is a diameter. **AD** is $\perp$ to the tangent at **B**. Show that **AB** bisects the $\angle$ **DAC**.

2. Two circles intersect at **A** and **B**. Any point **P** on the circumference of one circle is joined to **A** and **B** and the joining lines are produced to meet the circumference of the other circle at **C**, **D**. Show that **CD** is $\parallel$ to the tangent at **P**.

3. **LMN** is a $\triangle$. Show how to draw the tangent at **L** to the circumscribed circle, without finding the centre of this circle.

4. **If either of the $\angle$s which a st. line, drawn through one end of a chord of a circle, makes with the chord is equal to the $\angle$ in the segment on the other side of the chord, the st. line is a tangent.** *(Converse of III—15.)*

5. The tangent at a point **P** on a circle meets the chord **MN** produced through **N**, at **Q**. Prove $\angle$ **Q** = $\angle$ **PNM** − $\angle$ **PMN**.

6. A tangent drawn $\parallel$ to a chord of a circle bisects the arc cut off by the chord.

7. **FGE, HKE** are two circles, and **FEH, GEK** two st. lines. Prove that **FG, KH** meet at an $\angle$ which $=$ the $\angle$ between the tangents to the circles at **E**.

8. **G** is the middle point of an arc **EGF** of a circle. Show that **G** is equidistant from the chord **EF** and the tangent at **E**.

9. A st. line **EF** is trisected in **G, H**, and an equilateral $\triangle$ **PGH** is described on **GH**. Show that the circle **FGP** touches **EP**.

10. **D, E, F** are respectively the points of contact of the sides **MN, NL, LM** of a $\triangle$ circumscribed about a circle. **DG, EH** are respectively $\perp$ **EF, DF**. Prove **GH** $\parallel$ **LM**.

11. The tangent at **L** to the circumscribed circle of $\triangle$ **LMN** meets **MN** produced at **D**, and the internal and external bisectors of the $\angle$ **MLN** meet **MN** at **E, F** respectively. Prove that **D** is the middle point of **EF**.

12. **GEF, HEF** are two circles and **GEH** is a st. line. The tangents at **G, H** meet at **K**. Show that **K, G, F, H** are concyclic.

13. Points **P, Q** are taken on two st. lines **LM, LN** so that **LP** $+$ **LQ** $=$ a given st. line. Prove that the circle **PLQ** passes through a second fixed point.

14. **E, F, G, H** are the points of contact of the sides **XY, YZ, ZV, VX** of a quadrilateral circumscribed about a circle. If **X, Y, Z, V** are concyclic, show that **EG** $\perp$ **FH**.

15. **XYZV** is a quadrilateral inscribed in a circle, and **XZ, YV** cut at **E**. Prove that the tangent at **E** to the circle **XEY** is $\parallel$ **ZV**.

16. **F** is the point of contact of a tangent **EF** to the circle **FGH**. **GK** drawn $\parallel$ **EF** meets **FH**, or **FH** produced,

at K. Show that the circle through G, K, H touches FG at G.

17. If from an external point P a tangent PT and a secant PMN be drawn to a circle, the $\triangle$s PTM, PNT are similar.

18. Use III—15 to prove that the tangents drawn to a circle from an external point are equal.

19. From an external point T a tangent TR and a secant TQP through the centre are drawn to a circle. Prove that $\angle$ T + 2 $\angle$ TRQ = a rt. $\angle$.

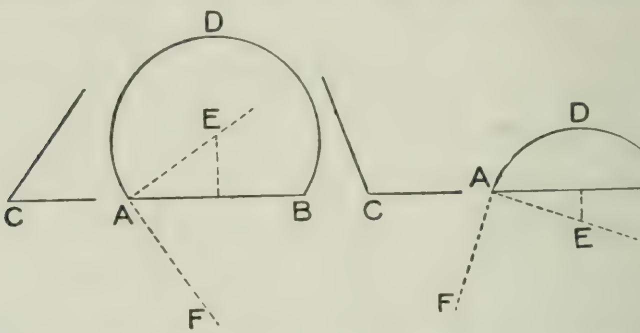
20. The tangents OT, OS from a fixed point O to a given circle contain an $\angle$ of x degrees. A third tangent is drawn to the circle at any point on the minor arc TS. Show that the portion of this tangent intercepted by OT and OS subtends an $\angle$ of $(90 - \frac{x}{2})$ degrees at the centre.

Show that if the moving point be taken on the major arc TS, the $\angle$ at the centre will be $(90 + \frac{x}{2})$ degrees.

Constructions

Problem 4

On a given straight line to construct a segment. containing an angle equal to a given angle.



Let AB be the given st. line, and C the given $\angle$.

Construction.—Make $\angle$ BAF = $\angle$ C.

Draw AE $\perp$ AF.

Draw the right bisector of AB and produce it to cut AE at E.

$\because$ E is in the right bisector of AB, it is equidistant from A and B. (I—22, p. 78.)

With centre E and radius EC describe the arc ADB.

ADB is the required arc.

Proof.—$\because$ AF is $\perp$ AE,

$\therefore$ AF is a tangent to the circle ADB. (III—14, Cor. 1, p. 171.)

$\because$ AB is a chord drawn from the point of contact of the tangent AF,

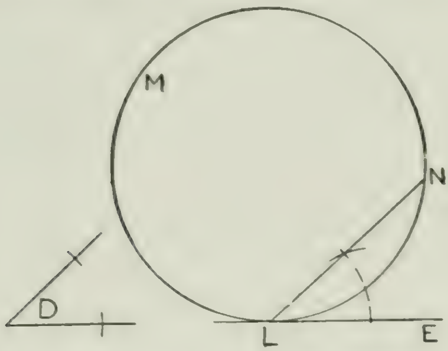
$\therefore$ $\angle$ in segment ADB = $\angle$ FAB. (III—15, p. 177.)

But, $\angle$ FAB = $\angle$ C, (*Const.*)

$\therefore$ $\angle$ in segment ADB = $\angle$ C.

Problem 5

From a given circle to cut off a segment containing an angle equal to a given angle.



Let LMN be the given circle, and D the given ∠.

Construction. — Draw a tangent LE to the given circle.

At L make the ∠ ELN = ∠ D.

LMN is the required segment.

Proof. — ∵ LE is a tangent, and LN a chord,

∴ ∠ in segment LMN = ∠ NLE.

(III—15, p. 177.)

But, ∠ NLE = ∠ D. (*Const.*)

∴ ∠ in segment LMN = ∠ D.

108.—Exercises

1. On st. lines each 4 cm. in length, describe segments containing ∠s of (*a*) 45°, (*b*) 150°, (*c*) 72°, (*d*) 116°. (*Use the protractor for* (*c*) *and* (*d*).)

2. On a given base construct an isosceles △ with a given vertical ∠.

3. Divide a circle into two segments such that the $\angle$ in one segment is (*a*) twice, (*b*) three times, (*c*) five times, (*d*) seven times the $\angle$ in the other segment.

4. Construct two $\triangle$s ABC_1, ABC_2 on the same base $AB = 4$ cm., having $\angle AC_1B = \angle AC_2B = 50°$, and $AC_1 = AC_2 = 5$ cm.

Prove that $\angle ABC_1 + \angle ABC_2 = 2$ rt. $\angle$s.

5. Construct a $\triangle$ LMN having $LM = 5$ cm., $\angle N = 110°$, and the median from $N = 2$ cm.

Measure the greatest and least values the median from N could have, with $LM = 5$ cm., and $\angle N = 110°$.

6. Construct a $\triangle$ having its base 5 cm., its vertical $\angle$ 70°, and its altitude 3 cm.

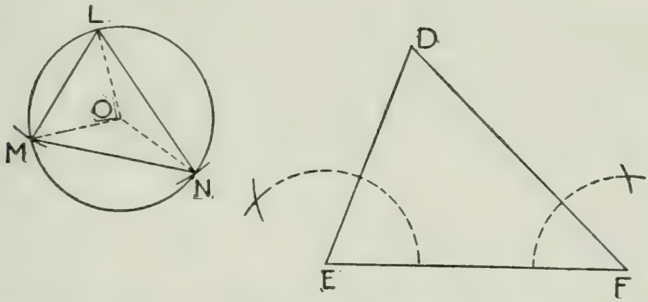
7. Construct a $\triangle$ XYZ, having $XY = 4$ cm., $\angle Z = 40°$, and $XZ + ZY = 10$ cm.

8. Construct a $\triangle$ XYZ, having $XY = 6$ cm., $\angle Z = 50°$ and $XZ - ZY = 4$ cm.

9. Through a given point draw a st. line to cut off from a given circle a segment containing an $\angle$ equal to a given $\angle$.

PROBLEM 6

In a given circle to inscribe a triangle similar to a given triangle.



Let LMN be the given circle, and DEF the given △.

Construction.—Draw a radius OL of the circle.

Make ∠ LON = 2 ∠ E, and ∠ LOM = ∠ 2 F.

Join LM, MN, NL.

LMN is the required △.

Join OM, ON.

Proof.—∵ ∠ LON at the centre and ∠ LMN at the circumference stand on the same arc.

∴ ∠ LON = 2 ∠ LMN, (III—6, p. 152.)

But ∠ LON = 2 ∠ E, (*Const.*)

∴ ∠ LMN = ∠ E.

Similarly ∠ LNM = ∠ F.

∵ ∠ LMN = ∠ E,

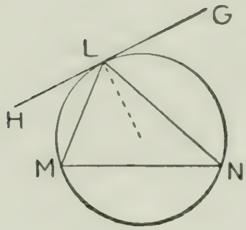
and ∠ LNM = ∠ F,

∴ ∠ MLN = ∠ D. (I—10, p. 45.)

and ∴ △ LMN ||| △ DEF.

109.—Exercises

1. Prove the following construction for inscribing a △ similar to a given △ **DEF** in the circle **LMN**.



Draw a tangent **HLG**. Make ∠ **GLN** = ∠ **E**, and ∠ **HLM** = ∠ **F**. Join **MN**.

2. Inscribe an equilateral △ in a given circle.

3. Inscribe a square in a given circle.

4. Inscribe a regular pentagon in a given circle. (Use protractor).

5. Inscribe a regular hexagon in a given circle. (Without protractor).

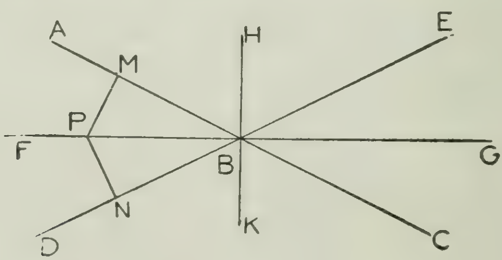
6. Inscribe a regular octagon in a given circle.

7. Two △s **LMN**, **DEF**, each similar to a given △ **GHK**, are inscribed in a given circle. Prove △ **LMN** ≡ △ **DEF**.

8. In a given circle inscribe a △ having its sides ∥ to the sides of a given △.

Problem 7

To find the locus of the centres of circles touching two given intersecting straight lines.



Let **ABC**, **DBE** be the two st. lines.

Construction.—Draw the bisectors **FBG, HBK** of the ∠s made by **AC** and **DE.**

These bisectors make up the required locus.

Proof.—Take a point **P** in either **FG** or **HK**, and draw **PM** ⊥ **AC**, **PN** ⊥ **DE.**

In △s **PMB, PNB,** $\begin{cases} \angle \text{PBM} = \angle \text{PBN}, \\ \angle \text{PMB} = \angle \text{PNB}, \\ \text{and PB is common,} \end{cases}$

∴ **PM = PN.** (I—14, p. 54.)

Hence, a circle described with centre **P** and radius **PM** will pass through **N.**

∵ ∠s at **M, N** are rt. ∠s,

∴ **AC, DE** are tangents to the circle.

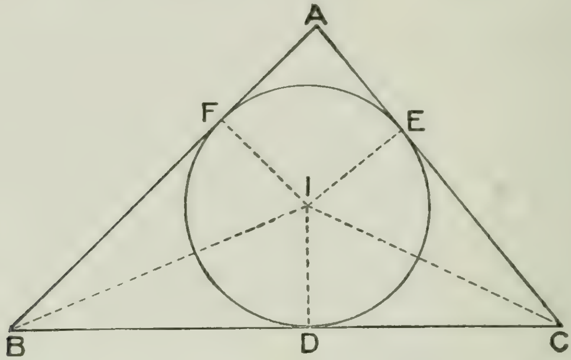
(III—14, Cor. 1, p. 171.)

110. **Definitions.**—When a circle is within a triangle, and the three sides of the triangle are tangents to the circle, the circle is said to be **inscribed in the triangle,** and is called the **inscribed circle of the triangle.**

When a circle lies without a triangle, and touches one side and the other two sides produced, the circle is called an **escribed circle of the triangle.**

Problem 8

To inscribe a circle in a given triangle.



Let ABC be the given $\triangle$.

Bisect $\angle$s B and C and produce the bisectors to meet at I.

Draw ID, IE, IF, $\perp$ BC, CA, AB respectively.

In $\triangle$s BID, BIF, $\begin{cases} \angle \text{IBD} = \angle \text{IBF}, \\ \angle \text{IDB} = \angle \text{IFB}, \\ \text{IB is common,} \end{cases}$

$\therefore$ ID = IF. (I—14, p. 54.)

Similarly, ID = IE.

$\therefore$ a circle described with centre I and radius ID will pass through E and F.

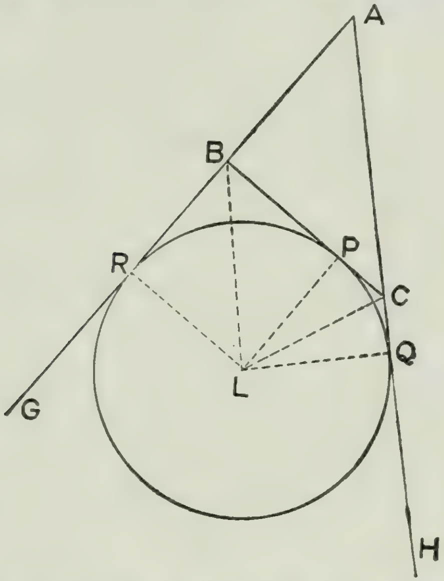
And $\because$ the $\angle$s at D, E and F are rt. $\angle$s,

$\therefore$ the circle will touch BC, CA and AB.

(III—14, Cor. 1, p. 171.)

PROBLEM 9

To draw an escribed circle of a given triangle.



Let **ABC** be a given $\triangle$ having **AB**, **AC** produced to **G**, **H**.

It is required to describe a circle touching the side **BC** and the two sides **AB**, **AC** produced.

Bisect $\angle$s **GBC**, **HCB** and let the bisectors meet at **L**. Draw $\perp$s **LP**, **LQ**, **LR** to **BC**, **CH**, **BG** respectively.

In $\triangle$s **LBP**, **LBR**, $\left\{\begin{array}{l}\angle \text{PBL} = \angle \text{RBL}, \\ \angle \text{LPB} = \angle \text{LRB}, \\ \text{LB is common,}\end{array}\right.$

$\therefore$ **LP** = **LR**. (I—14, p. 54.)

Similarly **LP** = **LQ**.

$\therefore$ a circle described with centre **L** and radius **LP** will pass through **R** and **Q**.

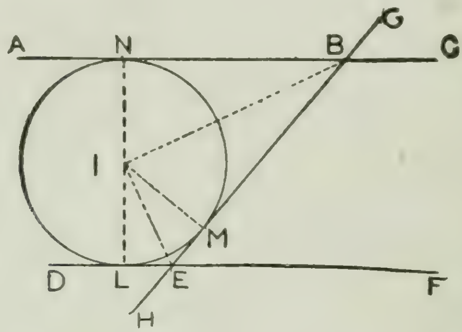
$\because$ the $\angle$s at **P**, **Q** and **R** are rt. $\angle$s,

$\therefore$ the circle will touch **BC**, and **CA** and **AB** produced. (III—14, Cor. 1, p. 171.)

Problem 10

To describe a circle to touch three given straight lines.

(*a*) If two of the lines are ∥ to each other, and the third cuts them, two circles may be drawn to touch the three lines.



Let ABC, DEF and GBEH be the three lines of which AC ∥ DF.

Bisect ∠s ABE, BED, and produce the bisectors to meet at I.

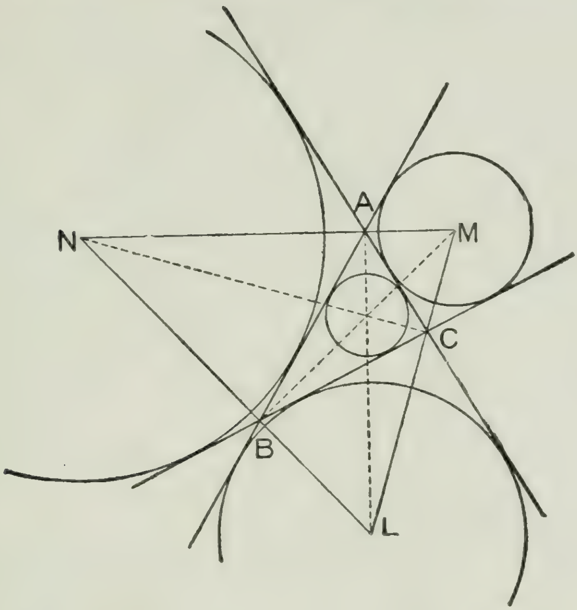
Draw IL, IM, IN ⊥ DE, EB, BA respectively.

As in problems 8 and 9 it may be shown that a circle described with centre I and radius IL will touch DE, EB and BA.

Similarly, a circle may be described on the other side of BE to touch the three given st. lines.

(*b*) If the lines intersect each other forming a △, four circles may be drawn to touch the three lines.

Let ABC be the $\triangle$ formed by the lines.



Draw the inscribed circle and the three escribed circles of $\triangle$ ABC.

These four circles touch the three given st. lines.

111.—Exercises

1. Make an $\angle$ YXZ = 45°. Find a point P such that its distance from XY is 3 cm., and its distance from XZ is 4 cm.

2. Make an $\angle$ YXZ = 60°. Find a point P such that its distance from XY is 4 cm., and its distance from XZ is 5 cm.

3. **The bisectors of the $\angle$s of a $\triangle$ are concurrent.**

4. The bisectors of the exterior $\angle$s at two vertices of a $\triangle$ and the bisector of the interior $\angle$ at the third vertex are concurrent.

5. If a, b, c represent the numerical measures of the sides BC, CA, AB respectively of $\triangle$ ABC, and $s = \frac{1}{2}(a + b + c)$,

(*a*) AF $= s - a$, BD $= s - b$, CE $= s - c$, when D, E and F are the points of contact of BC, CA, AB with the inscribed circle. (Diagram of Problem 8.)

(*b*) AR $= s$, BP $= s - c$, CP $= s - b$, where R and P are the points of contact of AB produced and of BC with an escribed circle. (Diagram of Problem 9.)

(*c*) If r be the radius of the inscribed circle, $rs =$ the area of △ ABC.

(*d*) If r_1 be the radius of the escribed circle touching BC, $r_1 (s - a) =$ the area of △ ABC.

6. If the base and vertical ∠ of a △ be given, find the locus of the inscribed centre.

7. If the base and vertical ∠ of a △ be given, find the loci of the escribed centres.

8. L, M, N are the centres of the escribed circles of △ PQR. Show that the sides of △ LMN pass through the vertices of △ PQR.

9. If the centres of the escribed circles be joined, and the points of contact of the inscribed circle with the sides be joined, the △s thus formed are similar.

10. Construct a △ having given the base, the vertical ∠ and the radius of the inscribed circle.

11. Describe a circle cutting off three equal chords of given length from the sides of a given △.

12. An escribed circle of △ ABC touches BC at D and also touches AB and AC produced. The inscribed circle touches BC at E. Show that DE equals the difference of AB and AC.

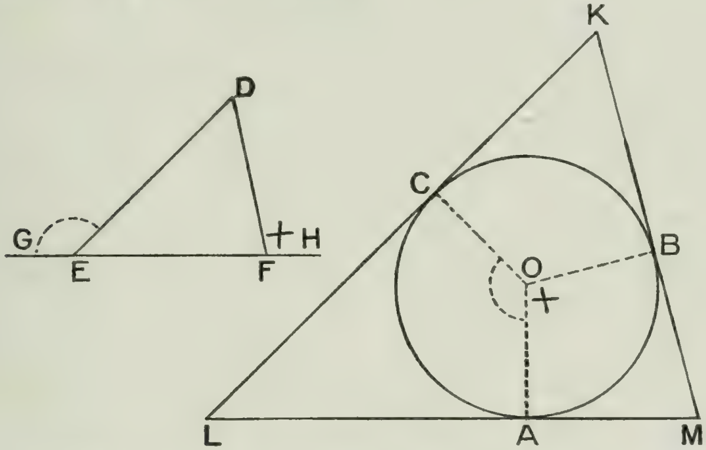
13. Circumscribe a square about a given circle.

14. Inscribe a circle in a given square.

15. Circumscribe a circle about a given square.

Problem 11

About a given circle to circumscribe a triangle similar to a given triangle.



Let ABC be the given circle and DEF the given $\triangle$.

Construction.—Produce EF to G and H.

Draw any radius OA of the circle, and at O make $\angle$ AOB = $\angle$ DFH, and $\angle$ AOC = $\angle$ DEG; and produce the arms to cut the circle at B, C.

At A, B, C draw tangents to the circle meeting at K, L and M.

KLM is the required $\triangle$.

Proof.—$\because$ $\angle$s MAO and MBO in the quadrilateral MBOA are rt. $\angle$s,

$\therefore$ $\angle$ M + $\angle$ AOB = 2 rt. $\angle$s.

= $\angle$ DFE + $\angle$ DFH.

But, $\angle$ AOB = $\angle$ DFH,

$\therefore$ $\angle$ M = $\angle$ DFE.

Similarly, $\angle$ L = $\angle$ DEF.

$\therefore$ $\angle$ L + $\angle$ M = $\angle$ DEF + $\angle$ DFE,

and $\therefore$ $\angle$ K = $\angle$ EDF. (I—10, p. 45.)

$\therefore$ $\triangle$ KLM ||| $\triangle$ DEF.

112.—Exercises

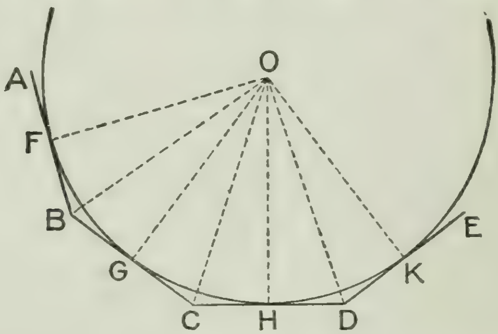
1. About a given circle circumscribe an equilateral △.

2. If two similar △s be circumscribed about the same circle, the △s are congruent.

3. Describe a △ LMN similar to a given △ and such that a given circle is touched by MN and by LM and LN produced.

Problem 12

To inscribe a circle in a given regular polygon.



Let AB, BC, CD, DE be four consecutive sides of a given regular polygon.

It is required to inscribe a circle in the polygon.

Bisect ∠s BCD, CDE and produce the bisectors to meet at O. Join OB. From O draw ⊥s OF, OG, OH, OK to AB, BC, CD, DE respectively.

In △s OCB, OCD, $\begin{cases} \text{BC} = \text{CD}, \\ \text{CO is common}, \\ \angle\ \text{OCB} = \angle\ \text{OCD}, \end{cases}$

∴ ∠ OBC = ∠ ODC. (I—2, p. 16.)

But ∠ ODC = $\frac{1}{2}$ ∠ CDE and ∠ ABC = ∠ CDE,

∴ ∠ OBC = $\frac{1}{2}$ ∠ ABC.

In the same manner it may be shown that if O be joined to all the vertices of the polygon the joining lines will bisect the $\angle$s at the vertices.

In $\triangle$s OCG, OCH, $\begin{cases} \angle \text{OCG} = \angle \text{OCH}, \\ \angle \text{OGC} = \angle \text{OHC}, \\ \text{OC is common}, \end{cases}$

$\therefore$ OG = OH. (I—14, p. 54.)

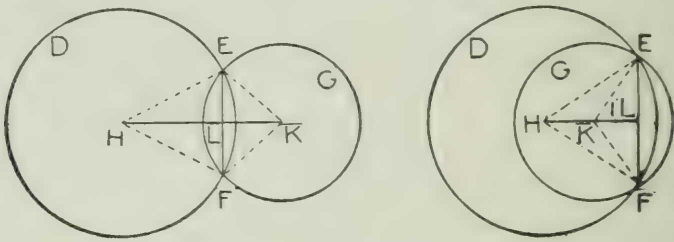
In the same manner it may be shown that the $\perp$s from O to all of the sides are equal to each other, and as the $\angle$s at F, G, H, etc., are rt. $\angle$s, a circle described with O as centre and OF as radius will touch each of the sides and be inscribed in the polygon.

Contact of Circles

113. **Definition.**—If two circles meet each other at one and only one point, they are said to **touch** each other at that point.

Theorem 16

If two circles touch each other, the straight line joining their centres passes through the point of contact.



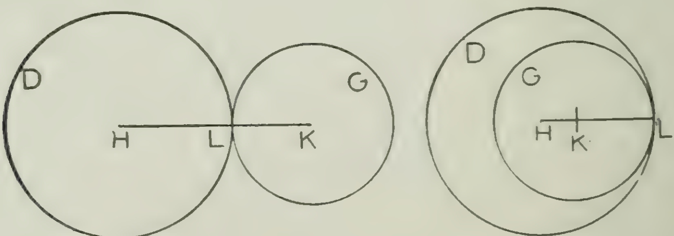
Let two circles **DEF, GEF,** of which the centres are **H, K** respectively, cut each other at **E, F.**

Join **HE, HF, KE, KF.**

∵ **HEF, KEF** are isosceles $\triangle$s on the same base **EF,**

∴ **HK** is an axis of symmetry of the quadrilateral **HEKF** and **E, F** are corresponding points. (I—5, p. 24.)

∴ **HK** bisects **EF.**

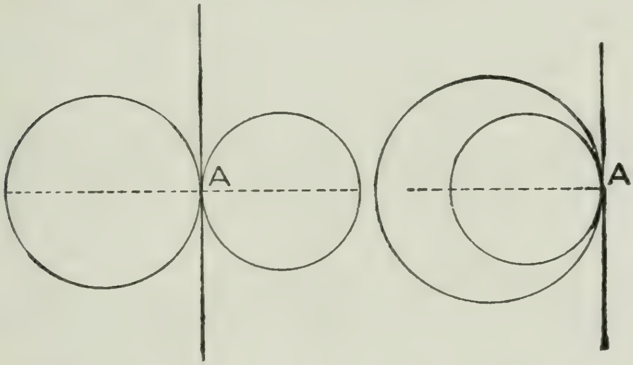


Let the circle **GEF** move so that the points **E, F** approach each other and finally coincide.

∵ L is the middle point of EF,

∴ L coincides with E and F, the circles touch at L, and the st. line HK passes through the point of contact L.

Cor. I.—**The straight line drawn from the point of contact perpendicular to the line of centres is a common tangent to the two circles.**



Definition.—If two circles which touch each other are on opposite sides of the common tangent at their point of contact, and consequently each circle outside the other, they are said to touch **externally**; if they are on the same side of the common tangent, and consequently one within the other, they are said to touch **internally.**

Cor. 2.—**If two circles touch externally, the distance between their centres is equal to the sum of their radii; and conversely.**

Cor. 3.—**If two circles touch internally, the distance between their centres is equal to the difference of their radii; and conversely.**

114.—Exercises

1. If the st. line joining the centres of two circles pass through a point common to the two circumferences, the circles touch each other at that point.

2. Find the locus of the centres of all circles which touch a given circle at a given point.

3. Draw three circles with radii 23, 32 and 43 mm. each of which touches the other two externally.

4. Draw a circle of radius 9 cm., and within it draw two circles of radii 3 cm. and 4 cm., to touch each other externally, and each of which touches the first circle internally.

5. Draw a circle of radius 85 mm., and within it draw two circles of radii 25 mm. and 35 mm., to touch each other externally, and each of which touches the first circle internally.

6. Draw a △ ABC with sides 5, 12 and 13 cm. Draw three circles, with centres A, B and C respectively, each of which touches the other two externally.

7. Construct the △ ABC, having a = 5 cm., b = 4 cm., and c = 3 cm. Draw three circles with centres A, B and C respectively, such that the circles with centres B and C touch externally, and each touches the circle with centre A internally.

8. Mark two points P and Q 10 cm. apart. With centres P and Q, and radii 4 cm. and 3 cm., describe two circles. Draw a circle of radius 5 cm. which touches each of the first two circles externally. Find the distance of the centre from PQ.

9. Describe a circle to pass through a given point, and touch a given circle at a given point.

10. If two circles touch each other, any st. line drawn through the point of contact will cut off segments that contain equal ∠s.

11. Two circles **ACO, BDO** touch, and through **O,** st. lines **AOB, COD** are drawn. Show that **AC** || **BD.**

12. If two || diameters be drawn in two circles which touch one another, the point of contact and an extremity of each diameter are in the same st. line.

13. Describe a circle which shall touch a given circle, have its centre in a given st. line, and pass through a given point in the st. line.

14. Describe three circles having their centres at three given points and touching each other in pairs. Show that there are four solutions.

15. Two circles touch a given st. line at two given points, and also touch each other; find the locus of their point of contact.

16. If through the point of contact of two touching circles a st. line be drawn cutting the circles again at two points, the radii drawn to these points are ||.

17. In a given semi-circle inscribe a circle having its radius equal to a given st. line.

18. Inscribe a circle in a given sector.

19. A circle of 2·5 cm. radius has its centre at a distance of 5 cm. from a given st. line. Describe four circles each of 4 cm. radius to touch both the circle and the st. line.

20. If **DE** be drawn || to the base **GH** of a △ **FGH** to meet **FG, FH** at **D, E** respectively, the circles described about the △s **FGH, FDE** touch each other at **F.**

21. Two circles with centres **P, Q** touch externally and a third circle is drawn, with centre **R,** which both the first circles touch internally. Prove that the perimeter of △ **PQR** = the diameter of the circle with centre **R.**

Miscellaneous Exercises

1. If two chords of a circle intersect at rt. $\angle$s, the sum of the squares on their segments is equal to the square on the diameter.

2. Find a point in the circumference of a given circle, the sum of the squares on whose distances from two given points may be a maximum or minimum.

3. **AOB, COD** are chords cutting at a point **O** within the circle. Show that $\angle$ **BOC** equals an $\angle$ at the circumference, subtended by an arc which is equal to the sum of the arcs subtending $\angle$s **BOC, AOD.**

4. Two chords **AB, CD** intersect at a point **O** without a circle. Show that $\angle$ **AOC** equals an $\angle$ at the circumference subtended by an arc which is equal to the difference of the two arcs **BD, AC** intercepted between **OBA** and **ODC.**

5. Two circles touch externally at **E**, and are cut by a st. line at **A, B, C, D.** Show that $\angle$ **AED** is supplementary to $\angle$ **BEC.**

6. If at a point of intersection of two circles the tangents drawn to the circles be at rt. $\angle$s, the st. line joining the points where these tangents meet the circles again, passes through the other point of intersection of the circles.

7. Find a point within a given $\triangle$ at which the three sides subtend equal $\angle$s. When is the solution possible?

8. Through one of the points of intersection of two given circles draw the greatest possible st. line terminated in the two circumferences.

9. Through one of the points of intersection of two given circles draw a st. line terminated in the two circumferences and equal to a given st. line.

10. Describe a circle of given radius to touch two given circles.

11. DEF is a st. line cutting BC, CA, AB, the sides of △ ABC, at D, E, F respectively. Show that the circles circumscribed about the △s AEF, BFD, CDE, ABC, all pass through one point.

12. Two circles touch each other at A and BAC is drawn terminated in the circumferences at B, C. Show that the tangents at B, C are ∥.

13. D, E, F are any points on the sides BC, CA, AB of △ ABC. Show that the circles circumscribed about the △s AFE, BDF, CED pass through a common point.

14. Two arcs stand on a common chord AB. P is any point on one arc and PA, PB cut the other arc at C, D. Show that the length of CD is constant.

15. ACB is an ∠ in a segment. The tangent at A is ∥ to the bisector of ∠ ACB and meets BC produced at D. Show that AD = AB.

16. Describe a circle of given radius to touch two given intersecting st. lines.

17. In the △ ABC, the bisector of ∠ A meets BC at D. O is the centre of a circle which touches AB at A and passes through D. Prove that OD ⊥ AC.

18. The st. line BC of given length moves so that B and C are respectively on two given fixed st. lines AX and AY. Prove that the circumcentre of △ ABC lies on the circumference of a circle with centre A.

19. ABC is an isosceles △ in which AB = AC. D is any point in BC. Show that the centre of the circle ABD is the same distance from AB that the centre of the circle ACD is from AC.

20. **E, F, G, H** are the points of contact of the sides of a quadrilateral **ABCD** circumscribed about a circle. Prove that the difference of two opposite $\angle$s of **ABCD** = twice the difference of two adjacent $\angle$s of **EFGH**.

21. **ABC** is a $\triangle$ in which **AX, BY** are $\perp$ **BC, CA** respectively. Prove that the tangent at **X** to the circle **CXY** passes through the middle point of **AB**; and the tangent at **C** to the same circle $\parallel$ **AB**.

22. The inscribed circle of $\triangle$ **ABC** touches **BC** at **D.** Prove that the circles inscribed in $\triangle$s **BAD, CAD** touch each other.

23. **O** is the circumcentre of the $\triangle$ **ABC**, and **AO, BO, CO** produced meet the circumference in **D, E, F.** Prove $\triangle$ **DEF** $\equiv$ $\triangle$ **ABC.**

24. **ABC** is a rt.-$\angle$d $\triangle$, **A** being the rt. $\angle$. Prove that **BC** = the difference between the radius of the inscribed circle and the radius of the circle which touches **BC** and the other two sides produced.

25. Describe two circles to touch two given circles, the point of contact with one of these given circles being given.

26. Circles through two fixed points **A** and **B** intersect fixed st. lines, which terminate at **A** and are equally inclined to **AB** on opposite sides of it, in the points **L, M.** Prove that **AL** + **AM** is constant.

27. **AB** is a diameter and **CD** a chord of a given circle. **AX** and **BY** are both $\perp$ **CD.** Prove that **CX** = **DY.**

28. Through a fixed point **A** on a circle any chord **AB** is drawn and produced to **C** making **BC** = **AB.** Find the locus of **C.**

29. Construct a △ having given the base, the vertical ∠, and the length of the median drawn from one end of the base.

30. If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, a circle may be inscribed in the quadrilateral.

31. Construct a △ having given the vertical ∠, the base, and the point where the bisector of the vertical ∠ cuts the base.

32. From the ends of a diameter **BC** of a circle, ∥ chords **BE, CF** are drawn, meeting the circle again in **E** and **F.** Prove that **EF** is a diameter.

33. **ACFB** and **ADEB** are fixed circles; **CAD, CBE** and **DBF** are st. lines. Prove that **CF** and **DE** meet at a constant ∠.

34. **A, B, C, D** are four points in order on the circumference of a circle, and the arc **AB** = the arc **CD.** If **AC** and **BD** cut at **E,** the chord which bisects ∠s **AEB, CED** is itself bisected at **E.**

35. **AB, AC** are tangents at **B, C** to a circle, and **D** is the middle point of the minor arc **BC.** Prove that **D** is the centre of the inscribed circle of the △ **ABC.**

36. Construct an equilateral △ whose side is of given length so that its vertices may be on the sides of a given equilateral △.

37. **D, E, F** are the points of contact of the sides **BC, CA, AB** of a △ **ABC** with its inscribed circle. **FK** is ⊥ **DE,** and **EH** is ⊥ **FD.** Prove **HK** ∥ **BC.**

38. Tangents are drawn from a given point to a system of concentric circles. Find the locus of their points of contact.

39. From a given point A without a given circle draw a secant ABC such that AB = BC.

40. EF is a fixed chord of a given circle, P any point on its circumference. EM $\perp$ FP and FN $\perp$ EP. Find the locus of the middle point of MN.

41. K is the middle point of a chord PQ in a circle of which O is the centre. LKM is a chord. Tangents at L, M meet PQ produced at G, H respectively. Prove $\triangle$ OGL $\equiv$ $\triangle$ OHM.

42. LM is the diameter of the semi-circle LNM in which arc LN $>$ arc NM, and ND $\perp$ LM. A circle inscribed in the figure bounded by ND, DM and the arc NM touches DM at E. Show that LE = LN; and hence give a construction for inscribing the circle.

43. GK is a diameter and O the centre of a circle. A tangent KD = KO. From O a $\perp$ OE is drawn to GD. KE is joined and produced to meet the circumference in F. Prove that FE = FG.

44. LPM and LQRM are two given segments on the same chord LM. If P moves on the arc LPM such that LQP and MRP are st. lines, the length of QR is constant.

45. EFP, EFRS are two circles and PFR, PES are st. lines. O is the centre of the circle EFP. Prove that PO $\perp$ RS.

46. E, F are fixed points on the circles EPD, FQD, and PDQ is a variable st. line. PE, QF intersect at R. Find the locus of R.

47. The circle PEGF passes through the centre G of the circle QEF, and P, E, Q are in a st. line. Prove that PQ = PF.

48. Through two points on a diameter equally distant from the centre of a circle, $\parallel$ chords are drawn, show that

these chords are the opposite sides of a rectangle inscribed in the circle.

49. If through the points of intersection of two circles any two ∥ st. lines be drawn and the ends joined towards the same parts, the figure so formed is a ∥gm.

50. Any two ∥ tangents are drawn, one to each of two given circles; a st. line is drawn through the points of contact, show that the tangents to the circles at the other points of intersection are also ∥.

51. The hypotenuse of a rt.-∠d △ is fixed and the other two sides are moveable, find the locus of the point of intersection of the bisectors of the acute ∠s of the △.

52. From the middle point **L** of the arc **MLN** of a circle two chords are drawn cutting the chord **MN** and the circumference. Show that the four points of intersection are concyclic.

53. If from one end of a diameter of a circle, two st. lines be drawn to the tangent at the other end of the diameter, the four points of intersection—with the circle, and with the tangent—are concyclic.

54. **ABC** is a diameter of a circle, **B** being the centre. **AD** is a chord, and **BE** ⊥ to **AC** cutting the chord at **E**. Show that **BCDE** is a cyclic quadrilateral; and that the circles described about **ABE** and the quadrilateral **BCDE**, are equal.

55. Two circles intersect at **A** and **B**. From **A** two chords **AC** and **AE** are drawn one in each circle making equal ∠s with **AB**, st. lines **CBD** and **EBF** are drawn to cut the circles at **D** and **F**, prove **C, F, D, E** concyclic; also prove △s **FCA** and **DEA** similar.

56. **ABC** is a △ and any circle is drawn passing through **B**, and cutting **BC** at **D** and **AB** at **F**; another circle is

drawn passing through C and D and intersecting the former circle at E and AC at G. Prove A, F, E, G are concyclic.

57. If two equal circles intersect, the four tangents at the points of intersection form a rhombus.

58. If two equal circles cut, and at G, one of the points of intersection, chords be drawn in each circle, to touch the other circle, these chords are equal.

59. Two equal circles, centres O and P, touch externally at S, SQ and SR are drawn $\perp$ to each other cutting the circumferences at Q and R respectively. Show that O, P, Q and R are the vertices of a ||gm.

60. AB, CD, and EF are || chords in a circle, prove that the $\triangle$s ACE and BDF are congruent; also ACF and BDE; also ADF and BCE.

61. On the circumference of a circle are two fixed points which are joined to a moveable point either inside or outside the circle. If these lines intercept a constant arc, find the locus of the point.

62. KL is any chord of a circle and H the middle point of one of the arcs, any st. line HED cuts KL at E and the circumference at D. Show that HL is a tangent to the circle about LED, and HK a tangent to that about KED.

63. Two circles intersect at E and F. From any point P on the circumference of one of them st. lines PE and PF are drawn to meet the circumference of the other at Q and R, show that the length of the straight line QR is constant. [Take P both on the major arc and on the minor arc.]

64. HKL is a $\triangle$ having $\angle$ H acute; on KL as diameter a circle, centre O, is described cutting HK at D and HL at E. Show that $\angle$ ODE = $\angle$ H.

65. **P** is a point external to two concentric circles whose centre is **O**, **PQ** is a tangent to the outer circle and **PR** and **PS** are tangents to the inner circle. Show that $\angle$ **RQS** is bisected by **QO**.

66. If the extremities of two $\parallel$ diameters in two circles be joined by a st. line which cuts the circles, the tangents at the points of intersection are $\parallel$. Show that this is true for the four cases that arise.

67. **KLMN** is a $\parallel$gm, through **L** and **N** two $\parallel$ st. lines are drawn cutting **MN** at **F** and **KL** at **E**, show that the circles described about the $\triangle$s **KNE** and **LMF** are equal.

68. **EFGH** is a quadrilateral having **EF** $\parallel$ **HG** and **EH** = **FG**. From **E** a st. line **EK** is drawn $\parallel$ **FG** meeting **HG** at **K**. Show that circles described about the $\triangle$s **EHG**, **EKG** are equal.

69. From any point **P** on the circumference of a circle **PD**, **PE** and **PF** are perpendiculars to a chord **QR**, and to the tangents **QT** and **RT**. Show that the $\triangle$s **PED** and **PFD** are similar.

70. A quadrilateral having two $\parallel$ sides is described about a circle. Show that the st. line drawn through the centre $\parallel$ to the $\parallel$ sides and terminated by the nonparallel sides is one quarter of the perimeter of the quadrilateral.

71. **CD** is a diameter of a circle centre **O**; chords **CF** and **DG** intersect within the circle at **E**. Show that **OF** is a tangent to the circle passing through **F**, **G** and **E**.

72. **EF** is a chord of a circle and **EP** a tangent; a st. line **PG** $\parallel$ to **EF** meets the circle at **G**; prove that the $\triangle$s **EFG** and **EPG** are similar.

73. The diagonals of a quadrilateral are $\perp$; show that the st. lines joining the feet of the perpendiculars from

the intersection of the diagonals on the sides form a cyclic quadrilateral.

74. Two chords of a circle intersect at rt. ∠s and tangents are drawn to the circle from the extremities of the chords; show that the resulting quadrilateral is cyclic.

75. A quadrilateral is described about a circle and its vertices are joined to the centre cutting the circumference in four points. Show that the diagonals of the quadrilateral formed by joining these four points are ⊥.

76. **DEF** is a △ inscribed in a circle whose centre is **O.** On **EF** any arc of a circle is described and **ED, FD,** or these lines produced, meet the arc at **P, Q.** Show that **OD,** or **OD** produced, cuts **PQ** at rt. ∠s.

77. **PQRS** is a ||gm and the diagonals intersect at **E.** Show that the circles described about **PES** and **QER** touch each other; and likewise those about **PEQ** and **RES.**

78. Two equal circles intersect at **E** and **F**; with centre **E** and radius **EF** a circle is described cutting the circles at **G** and **H.** Show that **FG** and **FH** are tangents to the equal circles.

79. If from any point on the circumference of a circle perpendiculars be drawn to two fixed diameters, the line joining their feet is of constant length.

80. From the extremities of the diameter of a circle perpendiculars are drawn to any chord. Show that the centre is equally distant from the feet of the perpendiculars.

81. **EF** and **GH** are || chords in a circle, **F** and **H** being towards the same parts; a point **K** is taken on the circumference such that **GF** bisects ∠ **HGK.** Prove **GK** = **EF.**

82. Two circles intersect at **D** and **E,** and **KEL** and **PEQ** are two chords terminated by the circumferences. Show that the △s **DKP** and **DLQ** are similar.

83. If from two points outside a circle, equally distant from the centre and situated on a diameter produced, tangents be drawn to the circle, the resulting quadrilateral is a rhombus.

84. If the arcs cut off by the sides of a quadrilateral inscribed in a circle be bisected and the opposite points be joined, these two lines shall be $\perp$. (*Note.—Use Ex. 3.*)

85. **PQ** is a fixed st. line and **PM**, **QN** are any two $\parallel$ st. lines, **M** and **N** being towards the same parts. The $\angle$s **MPQ** and **NQP** are bisected by **PR** and **QR**. Find the locus of **R**.

86. If the $\angle$s of a $\triangle$ inscribed in a circle be bisected by lines which meet the circumference, and a new $\triangle$ be formed by joining these points on the circumference, its sides shall be $\perp$ to the bisectors.

87. If two circles touch each other internally, and a st. line be drawn $\parallel$ to the tangent at the point of contact, the two intercepts between the circumferences subtend equal $\angle$s at the point of contact.

88. **ABC** is a $\triangle$ inscribed in a circle and **BA** is produced to **E**; **D** is any point in **AE**; circles are described through **B, C, D** and through **B, C, E**; **CFDG** cuts the circles **ABC, EBC** in **F** and **G**. Prove that $\triangle$s **ADF** and **DEG** are similar.

89. Draw a tangent to a circle which shall bisect a given $\parallel$gm which is outside the circle.

90. In a given circle draw a chord of fixed length which shall be bisected by a given chord.

91. In a given circle draw a chord which shall pass through a given point and be bisected by a given chord. How many such chords can be drawn?

92. Describe a circle with given radius to touch a given st. line and have its centre in another given st. line.

93. Describe a circle of given radius to pass through a given point and touch a given st. line.

94. Describe a circle to touch a given circle at a given point and a given st. line.

95. In a given st. line find a point such that the st. lines joining it to two given points may be (*a*) $\perp$s, (*b*) make a given $\angle$ with each other.

96. Describe a circle of given radius to touch a given circle and a given st. line.

97. Describe a circle to touch a given circle and a given st. line at a given point.

98. Inscribe in a given circle a $\triangle$ one of whose sides shall be equal to a given st. line, and such that the other two may pass through two given points respectively.

99. Place a chord PQ in a circle so that it will pass through a given point O within the circle, and such that the difference between OP and OQ may be equal to a given st. line.

100. Find two points on the circumference of a given circle which shall be concyclic with two given points P and Q outside the circle.

101. Describe a square (EFGH) having given the point F and two points P and Q in the sides FE and EH respectively.

102. Describe a square (EFGH) having given the point G and two points P and Q in the sides FE and EH respectively.

103. Describe a square so that its sides shall pass respectively through four given points.

104. If three circles touch externally at P, Q, R and PQ and PR meet the circumference of QR at D and E, then DE is a diameter, and is || to the line joining the centres of the other two circles.

105. Two equal circles intersect so that the tangents at one of the points of intersection are ⊥s. Show that the square on the diameter is twice the square on the common chord.

106. LMN is a rt.-∠d △, L being the rt. ∠, and LD is ⊥ to MN. Show that LM is a tangent to the circle LDN.

107. PQ is a tangent to a circle and PRS a secant passing through the centre, QN is ⊥ to PS. Show that QR bisects ∠ PQN.

108. LMN is a △ inscribed in a circle whose centre is O. Show that the radius OL makes the same ∠ with LM that the ⊥ from L to MN makes with LN.

109. If two chords of a circle be ⊥, the sum of one pair of opposite intercepted arcs is equal to the sum of the other pair.

110. On the sides of a quadrilateral as diameters circles are described. Show that the common chords of every adjacent pair of circles is || to the common chord of the remaining pair.

111. Two equal circles are so situated that the distance between their nearest points is less than the diameter of either circle. Show how to draw a st. line cutting them so as to be trisected by the circumferences.

112. LMN is a △ and D, E, F are the middle points of MN, NL and LM respectively; if LP is the perpendicular on MN, show that D, P, E, F are concyclic.

113. QR is a fixed chord of a circle and P a moveable point on the circumference. Find the locus of the intersection of the diagonals of the ||gm having PQ and QR for adjacent sides.

114. If a quadrilateral having two || sides is inscribed in a circle, show that the four perpendiculars from the middle point of an arc cut off by one of the || sides, to the two diagonals and to the nonparallel sides, are equal.

115. ABCD and A′B′C′D′ are any rectangles inscribed in two concentric circles respectively. P is on the circumference of the former circle and P′ on the latter. Prove $PA'^2 + PB'^2 + PC'^2 + PD'^2 = P'A^2 + P'B^2 + P'C^2 + P'D^2$.

116. A point Y is taken in a radius of a circle whose centre is O; on OY as base an isosceles △ XOY is described having X on the circumference; XO and XY are produced to meet the circumference at D and Z respectively, and E is the point between D and Z where the perpendicular from O to OY cuts the circle. Show that the arc DE is one-third of arc EZ.

BOOK IV

Ratio and Proportion

115. **Definitions.**—The **ratio** of one magnitude to another *of the same kind* is the number of times that the first contains the second; or it is the part, or fraction, that the first magnitude is of the second.

Thus the ratio of one magnitude to another is the same as the measure of the first when the second is taken as the unit.

If a st. line is 5 cm. in length, the ratio of its length to the length of one centimetre is 5, that is, the st. line is to one centimetre as 5 is to 1.

If two st. lines A, B are respectively 8 inches and 3 inches in length, then the ratio of A to B is 8 to 3.

The ratio of one magnitude A to another B is written either $\frac{A}{B}$, or A : B.

When the form $\frac{A}{B}$ is used, the upper magnitude is called the **numerator**, and the lower the **denominator**; and when the form A : B is used, the first magnitude is called the **antecedent**, and the second the **consequent**. The two magnitudes are called the **terms** of the ratio.

116. **Definitions.** — **Proportion** is the equality of ratios, *i.e.*, when two ratios are equal to each other, the four magnitudes are said to be in proportion.

The equality of the ratios of K to L and of M to N may be written in any one of the three forms:—
$\frac{K}{L} = \frac{M}{N}$, K : L = M : N or K : L :: M : N; and is read "K is to L as M is to N."

The four magnitudes in a proportion are called **proportionals.**

The first and last are called the **extremes,** and the second and third are called the **means.**

The first two magnitudes of a proportion must be of the same kind, and the last two must be of the same kind; but the first two need not be of the same kind as the last two. Thus in the proportion $\frac{D}{E} = \frac{F}{H}$, D and E may be lengths of lines, while F and H are areas.

117. **Definitions.**—Three magnitudes are said to be **in continued proportion,** or in **geometric progression,** when the ratio of the first to the second equals the ratio of the second to the third.

Three magnitudes L, M, N, of the same kind, are in continued proportion, if $\frac{L}{M} = \frac{M}{N}$.

e. g.:— L $= 4$ cm., M $= 6$ cm., N $= 9$ cm.

The second magnitude of a continued proportion is called the **mean proportional,** or **geometric mean,** of the other two.

118. Two magnitudes of the same kind are **commensurable** when each contains some common measure an integral number of times.

Two magnitudes of the same kind are **incommensurable** when there is no common measure, however small, contained in each of them an integral number of times.

The diagonal and side of a square are incommensurable; the ratio of the diagonal to the side being $\sqrt{2} : 1$.

The side of an equilateral triangle and the perpendicular from a vertex to the opposite side are incommensurable; the ratio of a side to the perpendicular being $2 : \sqrt{3}$.

$\sqrt{2} = 1{\cdot}414$ nearly, and $\sqrt{3} = 1{\cdot}732$ nearly, but while these roots may be calculated to any required degree of accuracy they cannot be exactly found. Thus there is no straight line however short that is contained an integral number of times in both the diagonal and side of a square; or in both the side and altitude of an equilateral triangle.

The treatment of incommensurable magnitudes is too difficult for an elementary text-book, but as in algebra, the relations that are obtained in geometry for commensurable magnitudes hold good also for incommensurable magnitudes.

119. The following simple algebraic theorems are used in geometry:—

1. If $\frac{a}{b} = \frac{c}{d}$, $ad = bc$.

If four numbers be in proportion, the product of the extremes is equal to the product of the means.

2. If $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{c} = \frac{b}{d}$.

If four numbers be in proportion, the first is to the third as the second is to the fourth.

When a proportion is changed in this way the second proportion is said to be formed from the first by **alternation.**

In order that a given proportion may be changed by alternation, the four magnitudes must be of the same kind.

e.g.:— $\dfrac{2 \text{ ft.}}{5 \text{ ft.}} = \dfrac{4 \text{ ft.}}{10 \text{ ft.}}$ and, by alternation, $\dfrac{2 \text{ ft.}}{4 \text{ ft.}} = \dfrac{5 \text{ ft.}}{10 \text{ ft.}}$; but from the proportion $\dfrac{\text{st. line } \mathsf{D}}{\text{st. line } \mathsf{E}} = \dfrac{\text{area } \mathsf{F}}{\text{area } \mathsf{G}}$ another proportion cannot be inferred by alternation.

3. If $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{b}{a} = \dfrac{d}{c}$.

If four numbers be in proportion, the second is to the first as the fourth is to the third.

When a proportion is changed in this way the second proportion is said to be formed from the first by **inversion.**

4. If $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{a+b}{b} = \dfrac{c+d}{d}$.

If four numbers be in proportion, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

5. If $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{a-b}{b} = \dfrac{c-d}{d}$.

If four numbers be in proportion, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.

6. If $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

If four numbers be in proportion, the sum of the first and second terms is to the difference of the first and second terms as the sum of the third and fourth terms is to the difference of the third and fourth terms.

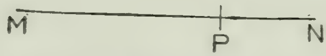
7. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ etc., then each of the equal fractions $= \frac{a + c + e + \text{etc.}}{b + d + f + \text{etc.}}$

If any number of ratios, the terms of which are all magnitudes of the same kind, be equal to each other, the sum of the numerators divided by the sum of the denominators equals each of the given ratios.

8. If $ad = bc$, $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{c} = \frac{b}{d}$.

If the product of two numbers be equal to the product of two other numbers, one factor of the first product is to a factor of the second product as the remaining factor of the second is to the remaining factor of the first.

120. If a given straight line **MN** be divided internally at a point **P**, the internal segments **PM**, **PN** are the distances from **P** to the ends of the given straight line.



Similarly, if a point **P** be taken in a given straight line **MN** produced, the distances from **P** to the ends of the given straight line, **PM**, **PN**, are called the **external segments of the straight**

line, or the given straight line is said to be divided **externally** at the point P.

121. There is only one point where a straight line MN is divided internally into segments MP, PN that have a given ratio $\frac{a}{b}$.

For, if possible, let it be divided internally at P and Q such that $\frac{MP}{PN}$ and $\frac{MQ}{QN}$ each equals $\frac{a}{b}$.

Then $$\frac{MP}{PN} = \frac{MQ}{QN}.$$

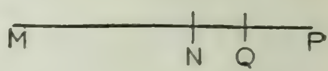
$$\therefore \frac{MP + PN}{PN} = \frac{MQ + QN}{QN}. \qquad (4, \S\, 119.)$$

i.e., $$\frac{MN}{PN} = \frac{MN}{QN}.$$

$$\therefore PN = QN.$$

and $\therefore$ Q coincides with P.

Similarly, there is only one point where a straight line MN is divided externally into segments MP, PN that have a given ratio $\frac{a}{b}$.



For, if possible, let it be divided externally at P and Q such that $\frac{MP}{PN}$ and $\frac{MQ}{QN}$ each equals $\frac{a}{b}$.

Then $$\frac{MP}{PN} = \frac{MQ}{QN}.$$

$$\therefore \frac{MP - PN}{PN} = \frac{MQ - QN}{QN}. \qquad (5, \S\, 119.)$$

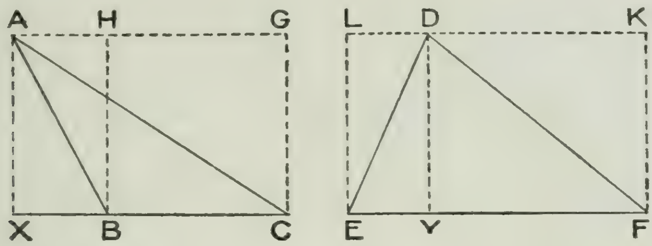
i.e., $$\frac{MN}{PN} = \frac{MN}{QN}.$$

$$\therefore PN = QN.$$

and $\therefore$ Q coincides with P.

Theorem 1

Triangles of the same altitude are to each other as their bases.



Hypothesis.—In $\triangle$s ABC, DEF; AX $\perp$ BC, DY $\perp$ EF and AX = DY.

To prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{BC}{EF}$.

Construction.—On BC and EF construct the rectangles HC and LF, having HB = AX and LE = DY.

Proof.—Let BC and EF contain a and b units of length respectively, and AX or DY contain c units.

$\triangle ABC = \frac{1}{2}\ HB.BC = \frac{1}{2}\ ca.$ (II—4, p. 100.)

$\triangle DEF = \frac{1}{2}\ LE.EF = \frac{1}{2}\ cb.$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}\ ca}{\frac{1}{2}\ cb} = \frac{a}{b} = \frac{BC}{EF}.$$

122.—Exercises

1. **$\triangle$s on equal bases are to each other as their altitudes.**

2. If two $\triangle$s are to each other as their bases, their altitudes must be equal.

3. **$\|$gms of equal altitudes are to each other as their bases.**

4. Construct a $\triangle$ equal to $\frac{5}{4}$ of a given $\triangle$.

5. Construct a $\|$gm equal to $\frac{5}{2}$ of a given $\|$gm.

6. **ABC, DEF** are two △s having **AB** = **DE** and ∠ **B** = ∠ **E**. Show that △ **ABC** : △ **DEF** = **BC** : **EF**.

7. The rectangle contained by two st. lines is a mean proportional between the squares on the lines.

8. If two equal △s be on opposite sides of the same base, the st. line joining their vertices is bisected by the common base, or the base produced.

9. The sum of the ⊥s from any point in the base of an isosceles △ to the two equal sides equals the ⊥ from either end of the base to the opposite side.

10. The difference of the ⊥s from any point in the base produced of an isosceles △ to the equal sides equals the ⊥ from either end of the base to the opposite side.

11. The sum of the ⊥s from any point within an equilateral △ to the three sides equals the ⊥ from any vertex to the opposite side.

12. If st. lines **AO, BO, CO** are drawn from the vertices of a △ **ABC** to any point **O** and **AO**, produced if necessary, cuts **BC** at **D**,

$$\frac{\triangle \mathrm{AOB}}{\triangle \mathrm{AOC}} = \frac{\mathrm{BD}}{\mathrm{DC}}.$$

13. In any △ **ABC**, **F** is the middle point of **AB**, **E** is the middle point of **AC**, and **BE**, **CF** intersect at **O**. Show that **AO** produced bisects **BC**; that is, **the medians of a △ are concurrent.**

14. **ABC** is a △ and **O** is any point. **AO, BO, CO**, produced if necessary cut **BC, CA, AB** at **D, E, F** respectively, a_1, a_2, b_1, b_2, c_1, c_2, are respectively the numerical measures of **BD, DC, CE, EA, AF, FB**. Show that $a_1\ b_1\ c_1 = a_2\ b_2\ c_2$. (This is known as Ceva's Theorem.)

15. The four △s into which a quadrilateral is divided by its diagonals are proportional.

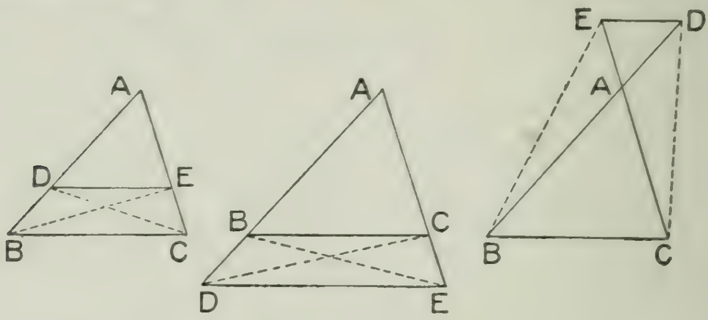
16. **DEF** is a $\triangle$; **G** is a point in **DE** such that **DG** = 3 **GE**, and **H** is a point in **DF** such that **FH** = 3 **HD**. Show that $\triangle$ **FGH** = 9 $\triangle$ **EGH**.

17. St. lines **DG, EH, FK** drawn from the vertices of $\triangle$ **DEF** to meet the opposite sides at **G, H, K** pass through a common point **O**. Prove that $\frac{DO}{DG} + \frac{EO}{EH} + \frac{FO}{FK} = 2$.

18. In $\triangle$ **DEF, G** is taken in side **EF** such that **EG** = 2 **GF**, and **H** is taken in side **FD** such that **FH** = 2 **HD**. **DG** and **EH** intersect at **O**. Prove that $\frac{\triangle DOH}{\triangle DEF} = \frac{1}{21}$.

Theorem 2

A straight line drawn parallel to the base of a triangle cuts the sides, or the sides produced, proportionally.



Hypothesis.—In $\triangle$ ABC, DE $\parallel$ BC.

To prove that $\dfrac{BD}{DA} = \dfrac{CE}{EA}$.

Construction.—Join BE and DC.

Proof.— $\because$ DE $\parallel$ BC,

$\therefore \triangle BDE = \triangle CDE$ (II—5, p. 101.)

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{\triangle CDE}{\triangle ADE}.$$

$\because$ $\triangle$s BDE, ADE have the same altitude, viz., the $\perp$ from E to AB,

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{BD}{DA}.$$ (IV—1, p. 219.)

In the same way,

$$\frac{\triangle CDE}{\triangle ADE} = \frac{CE}{EA}.$$

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}.$$

N.B.—By placing D *on* AB *and* E *on* AC *in all three figures the proof applies to all.*

Cor.—In the first figure,

$$\because \frac{BD}{DA} = \frac{CE}{EA}, \quad \therefore \frac{BD + DA}{DA} = \frac{CE + EA}{EA}. \quad \textit{by addition.}$$

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}.$$

$$\therefore \frac{AD}{AB} = \frac{AE}{AC} \quad \textit{by inverting.}$$

Again,

$$\because \frac{BD}{DA} = \frac{CE}{EA}, \quad \therefore \frac{DA}{BD} = \frac{EA}{CE}. \quad \textit{by inverting.}$$

$$\therefore \frac{DA + BD}{BD} = \frac{EA + CE}{CE}. \quad \textit{by addition.}$$

$$\therefore \frac{AB}{BD} = \frac{AC}{CE}.$$

$$\therefore \frac{BD}{AB} = \frac{CE}{AC}. \quad \textit{by inverting.}$$

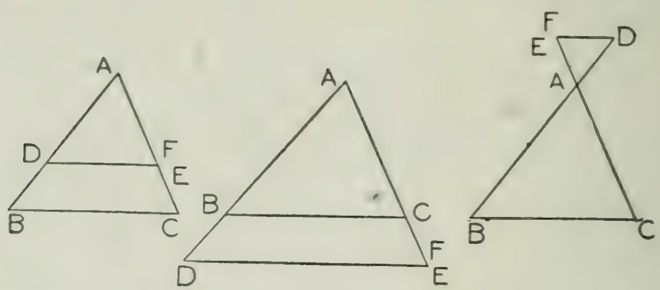
Similar proofs may be given for the second and third figures.

Thus we see that where a line is parallel to the base of a triangle we may form a proportion by taking the whole side or either of the segments, in any order, for the terms of the first ratio, provided we take the corresponding parts of the other side to form the terms of the other ratio in the proportion.

THEOREM 3

(Converse of Theorem 2)

If two sides of a triangle, or two sides produced, be divided proportionally, the straight line joining the points of section is parallel to the base.



Hypothesis.—In $\triangle$ ABC, $\frac{BD}{DA} = \frac{CE}{EA}$.

To prove that DE $\parallel$ BC.

Construction.—Draw DF $\parallel$ BC, to cut AC at F.

Proof.— $\because$ DF $\parallel$ BC,

$\therefore \frac{BD}{DA} = \frac{CF}{FA}$. (IV—2, p. 222.)

But $\frac{BD}{DA} = \frac{CE}{EA}$.

$\therefore \frac{CE}{EA} = \frac{CF}{FA}$,

And $\therefore$ E coincides with F.

$\therefore$ DE $\parallel$ BC.

123.—Exercises

1. The st. line drawn through the middle point of one side of a $\triangle$, and $\parallel$ to a second side bisects the third side.

2. The st. line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side.

3. If two sides of a quadrilateral be $\parallel$, any st. line drawn $\parallel$ to the $\parallel$ sides and cutting the other sides, will cut these other sides proportionally.

4. ABCD is a quadrilateral having AB ∥ DC. P, Q are points in AD, BC respectively such that AP : PD = BQ : QC. Show that PQ ∥ AB or DC.

5. **If two st. lines are cut by a series of ∥ st. lines, the intercepts on one are proportional to the corresponding intercepts on the other.**

6. D, E are points in AB, AC, the sides of △ ABC, such that DE ∥ BC; BE, CD meet at F. Show that △ ADF = △ AEF.

Show also that AF bisects DE and BC.

7. Through D, any point in the side BC of △ ABC, DE, DF are drawn ∥ AB, AC respectively and meeting AC, AB at E, F. Show that △ AEF is a mean proportional between △s FBD, EDC.

8. ACB, ADB are two △s on the same base AB. E is any point in AB. EF is ∥ AC and meets BC at F. EG is ∥ AD and meets BD at G. Prove FG ∥ CD.

9. D is a point in the side AB of △ ABC; DE is drawn ∥ BC and meets AC at E; EF is drawn ∥ AB and meets BC at F. Show that AD : DB = BF : FC.

10. From a given point M in the side DE of △ DEF, draw a st. line to meet DF produced at N so that MN is bisected by EF.

11. PQRS is a ∥gm, and from the diagonal PR equal lengths PK, RL are cut off. SK, SL when produced meet PQ, RQ respectively at E, F. Prove EF ∥ PR.

12. DEF is a △ in which K, M are points in the side DE and L, N are points in the side DF such that KL and MN are both ∥ EF. Find the locus of the intersection of KN and LM.

13. O any point within a quadrilateral PQRS is joined to the four vertices and in OP any point X is taken. XY

is drawn ∥ PQ to meet OQ at Y; YZ is drawn ∥ QR to meet OR at Z; and ZV is drawn ∥ RS to meet OS at V. Prove that XV ∥ PS.

14. O is a fixed point and P moves along a fixed st. line. Q is a point in OP, or in OP produced in either direction, such that OQ : QP is constant. Find the locus of Q.

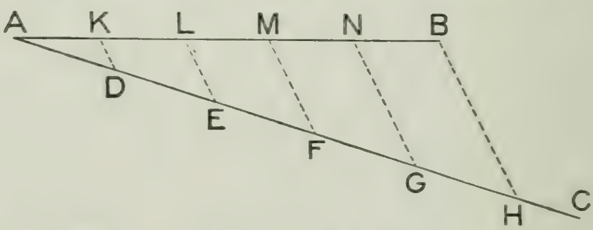
15. L is any point in the side DE of a △ DEF. From L a line drawn ∥ EF meets DF at M. From F a line drawn ∥ ME meets DE produced at N. Prove that DL : DE = DE : DN.

16. If from the vertex of a △ perpendiculars are drawn to the bisectors of the exterior ∠s at the base, the line joining the feet of the perpendiculars is ∥ the base.

Problem 1

To divide a given straight line into any number of equal parts.

(Alternative proof for I Prob. 8)



Let AB be the given straight line.

At A draw AC making any angle with AB and from AC cut off in succession the required number of equal parts. AD, DE, EF, FG, GH.

Join HB and through D, E, F, G draw lines ∥ BH cutting AB at K, L, M, N.

Then AK = KL = LM = MN = NB.

In $\triangle$ AEL, DK $\parallel$ EL,

$$\therefore \frac{AD}{DE} = \frac{AK}{KL}. \qquad \text{(IV—2, p. 222.)}$$

But AD = DE, $\therefore$ AK = KL.

In $\triangle$ AFM, EL $\parallel$ FM,

$$\therefore \frac{AE}{EF} = \frac{AL}{LM}.$$

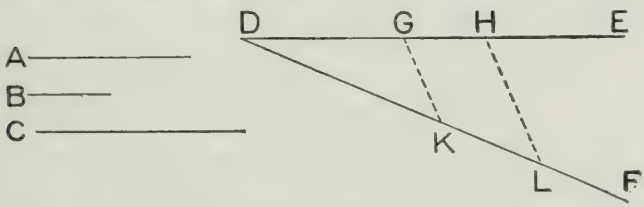
But AE = 2 EF, $\therefore$ AL = 2 LM

$\therefore$ LM = AK or KL.

In the same way it may be proved that AK = KL = LM = MN = NB.

PROBLEM 2

To find a fourth proportional to three given straight lines taken in a given order.



Let A, B, C be the three given st. lines.

From a point D draw two st. lines DE, DF.

Cut off DG = A, GH = B, DK = C.

Join GK. Through H draw HL $\parallel$ GK meeting DF in L.

Then KL is the required fourth proportional.

In $\triangle$ DHL, GK $\parallel$ HL

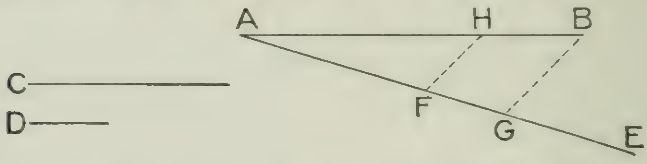
$$\therefore \frac{DG}{GH} = \frac{DK}{KL} \qquad \text{(IV—2, p. 222.)}$$

$$i.e., \quad \frac{A}{B} = \frac{C}{KL}$$

$\therefore$ KL is the required fourth proportional.

Problem 3

To divide a given straight line in a given ratio.



Let AB be the given st. line, and $\frac{C}{D}$ the given ratio.

Draw AE making any $\angle$ with AB.

On AE cut off AF = C, FG = D.

Join BG, and through F draw FH $\parallel$ GB.

In $\triangle$ ABG, $\because$ FH $\parallel$ GB,

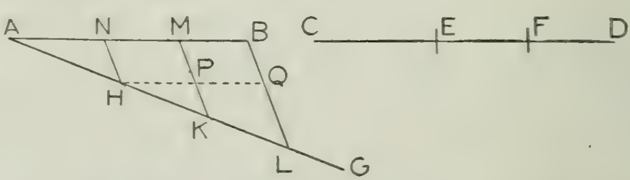
$$\therefore \frac{AH}{HB} = \frac{AF}{FG}. \quad \text{(IV--2, p. 222.)}$$

But AF = C, and FG = D,

$$\therefore \frac{AH}{HB} = \frac{C}{D}.$$

Problem 4

To divide a given straight line similarly to a given divided line.



Let AB be the given st. line, and CD the given line divided at E and F.

At A draw AG making any angle with AB.

From AG cut off AH = CE, HK = EF, KL = FD. Join BL. Through H, K draw HN, KM both $\parallel$ BL.

Then AB is divided at N and M similarly to CD. Through H draw HPQ $\parallel$ AB.

Proof.—In $\triangle$ AMK, NH $\parallel$ MK,

$$\therefore \frac{AN}{NM} = \frac{AH}{HK}. \qquad \text{(IV—2, p. 222.)}$$

In $\triangle$ HQL, PK $\parallel$ QL,

$$\therefore \frac{HP}{PQ} = \frac{HK}{KL}.$$

But HP = NM and PQ = MB,

$$\therefore \frac{NM}{MB} = \frac{HK}{KL}. \qquad \text{(I—20, p. 67.)}$$

$$\therefore \frac{AN}{NM} = \frac{CE}{EF} \text{ and } \frac{NM}{MB} = \frac{EF}{FD}.$$

Both these relations are contained in

$$\frac{AN}{CE} = \frac{NM}{EF} = \frac{MB}{FD}.$$

124.—Exercises

1. Divide the area of a given $\triangle$ into parts that are in the ratio of two given st. lines.

2. Divide the area of a $\parallel$gm into parts that are in the ratio of two given st. lines.

3. Find a third proportional to two given st. lines. Show how two third proportionals, one greater than either of the given st. lines and the other less than either, may be found.

4. **Divide a given st. line externally so that the ratio of the segments may equal the ratio of two given st. lines.**

5. BAC is a given $\angle$ and P is a given point. Through P draw a st. line DPE cutting AB at D and AC at E such that DP : PE equals the ratio of two given st. lines.

6. Divide a given st. line in the ratio 2 : 3 : 5.

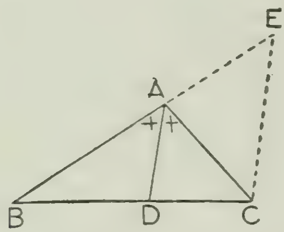
7. Construct a △ having its sides in the ratio 2 : 3 : 4, and its perimeter equal to a given st. line.

8. From a given point P outside the ∠ XOY draw a line meeting OX at Q and OY at R so that PQ : QR = a given ratio.

BISECTOR THEOREMS

THEOREM 4

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the segments of the base are proportional to the other sides of the triangle.



Hypothesis.—In △ ABC, AD bisects ∠ BAC.

To prove $\frac{BD}{DC} = \frac{BA}{AC}$.

Construction.—Through C draw CE ∥ AD to meet BA produced at E.

Proof.—AD ∥ EC, ∴ ∠ BAD = ∠ AEC, (I—9, p. 42.)

and ∠ DAC = ∠ ACE. (I—8, p. 40.)

But ∠ BAD = ∠ DAC, by hypothesis,

∴ ∠ AEC = ∠ ACE.

∴ AC = AE. (I—13, p. 52.)

In $\triangle$ EBC, AD $\parallel$ EC,

$$\therefore \frac{BD}{DC} = \frac{BA}{AE}. \qquad \text{(IV—2, p. 222.)}$$

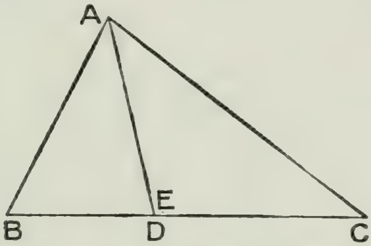
But AE = AC,

$$\therefore \frac{BD}{DC} = \frac{BA}{AC}.$$

THEOREM 5

(Converse of Theorem 4)

If the base of a triangle is divided internally into segments that are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the vertical angle.



Hypothesis. — In $\triangle$ ABC, $\frac{BD}{DC} = \frac{BA}{AC}$.

To prove that AD bisects $\angle$ BAC.

Construction.—Bisect $\angle$ BAC and let the bisector cut BC at E.

Proof.—$\because$ AE bisects $\angle$ BAC

$$\therefore \frac{BE}{EC} = \frac{BA}{AC}. \qquad \text{(IV—4, p. 230.)}$$

But, by hypothesis, $\frac{BD}{DC} = \frac{BA}{AC}$.

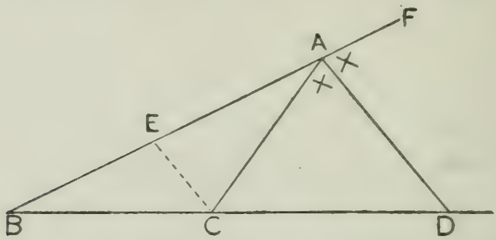
$$\therefore \frac{BE}{EC} = \frac{BD}{DC}.$$

$\therefore$ E and D coincide.

$\therefore$ AD bisects $\angle$ BAC.

Theorem 6

The bisector of the exterior vertical angle of a triangle divides the base externally into segments that are proportional to the sides of the triangle.



Hypothesis.—In $\triangle$ ABC, BA is produced to F.

$\angle$ FAC is bisected by AD which cuts BC produced at D.

To prove $$\frac{BD}{CD} = \frac{BA}{AC}.$$

Construction.—Through C draw CE $\parallel$ AD to meet AB at E.

Proof.—$\because$ EC $\parallel$ AD, $\therefore$ $\angle$ FAD = $\angle$ AEC. (I—9, p. 42.)

and $\angle$ DAC = $\angle$ ACE. (I—8, p. 40.)

But, by hypothesis, $\angle$ FAD = $\angle$ DAC.

$\therefore$ $\angle$ AEC = $\angle$ ACE.

$\therefore$ AC = AE. (I—13, p. 52.)

In $\triangle$ BAD, EC $\parallel$ AD,

$$\therefore \frac{BA}{AE} = \frac{BD}{DC}. \quad \text{(IV—2, Cor., p. 223.)}$$

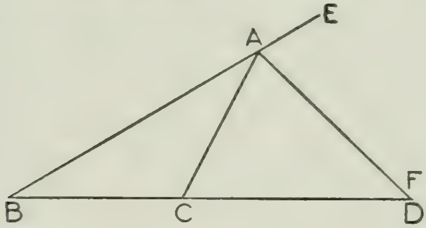
But AE = AC.

$$\therefore \frac{BA}{AC} = \frac{BD}{CD}.$$

Theorem 7

(Converse of Theorem 6)

If the base of a triangle is divided externally so that the segments of the base are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the exterior vertical angle.



Hypothesis.—In $\triangle$ ABC, $\frac{BD}{CD} = \frac{BA}{AC}$, and BA is produced to E.

To prove that AD bisects $\angle$ CAE.

Construction.—Bisect $\angle$ EAC by AF.

Proof.—$\because$ AF bisects exterior $\angle$ EAC,

$$\therefore \frac{BF}{CF} = \frac{BA}{AC}. \qquad \text{(IV—6, p. 232.)}$$

But, by hypothesis, $\frac{BD}{CD} = \frac{BA}{AC}$.

$$\therefore \frac{BF}{CF} = \frac{BD}{CD}.$$

$\therefore$ D and F coincide

$\therefore$ AD bisects $\angle$ EAC.

125.—Exercises

1. The sides of a $\triangle$ are 4 cm., 5 cm., 6 cm. Calculate the lengths of the segments of each side made by the bisector of the opposite $\angle$.

2. AD bisects $\angle$ A of $\triangle$ ABC and meets BC at D. Find BD and CD in terms of a, b, and c.

3. In △ **ABC**, $a = 7$, $b = 5$, $c = 3$. The bisectors of the exterior ∠s at **A, B, C** meet **BC, CA, AB** respectively at **D, E, F.** Calculate **BD, AE** and **AF.**

4. In △ **ABC**, the bisector of the exterior ∠ at **A** meets **BC** produced at **D.** Find **BD** and **CD** in terms of a, b and c.

5. If a st. line bisects both the vertical ∠ and the base of a △, the △ is isosceles.

6. The bisectors of the ∠s of a △ are concurrent. (Use IV—4 and 5.)

7. **AD** is a median of △ **ABC**; ∠s **ADB, ADC** are bisected by **DE, DF** meeting **AB, AC** at **E, F** respectively. Prove **EF** ∥ **BC.**

8. The bisectors of ∠s **A, B, C** in △ **ABC** meet **BC, CA, AB** at **D, E, F** respectively. Show that **AF.BD.CE** = **FB.DC.EA.**

9. If the bisectors of ∠s **A, C** in the quadrilateral **ABCD** meet in the diagonal **BD**, the bisectors of ∠s **B, D** meet in the diagonal **AC.**

10. If the bisectors of ∠s **ABC, ADC** in the quadrilateral **ABCD** meet at a point in **AC**, the bisectors of the exterior ∠s at **B** and **D** meet in **AC** produced.

11. If **O** is the centre of the inscribed circle of △ **DEF** and **DO** produced meets **EF** at **G**, prove that **DO : OG** = **ED + DF : EF.**

12. **PQ** is a chord of a circle ⊥ to a diameter **MN** and **D** is any point in **PQ.** The st. lines **MD, ND** meet the circle at **E, F** respectively. Prove that any two adjacent sides of the quadrilateral **PEQF** are in the same ratio as the other two.

13. The bisector of the vertical ∠ of a △ and the bisectors of the exterior ∠s at the base are concurrent.

14. One circle touches another internally at M. A chord PQ of the outer circle touches the inner circle at T. Prove that $\frac{PT}{TQ} = \frac{PM}{MQ}$.

15. LMN is a $\triangle$ in which LM $= 3$ LN. The bisector of $\angle$ L meets MN in D, and MX $\perp$ LD. Prove that LD $=$ DX.

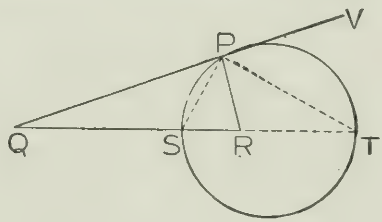
16. The $\angle$ A of $\triangle$ ABC is bisected by AD, which cuts the base at D, and O is the middle point of BC. Show that OD is to OB as the difference of AB and AC is to their sum.

17. The bisectors of the interior and exterior $\angle$s at the vertex of a $\triangle$ divide the base internally and externally in the same ratio.

18. **A point P moves so that the ratio of its distances from two fixed points Q, R is constant. Prove that the locus of P is a circle. (The Circle of Apollonius.)**

Divide QR internally at S and externally at T so that

$$\frac{QS}{SR} = \frac{QT}{TR} = \frac{PQ}{PR}.$$



Join PS, PT; and produce QP to V.

$$\because \frac{QS}{SR} = \frac{PQ}{PR}, \therefore \angle QPS = \angle SPR.$$

$$\because \frac{QT}{TR} = \frac{PQ}{PR}, \therefore \angle RPT = \angle TPV.$$

$$\therefore \angle SPT = QPS + \angle TPV$$
$$= \tfrac{1}{2} \text{ st. } \angle QPV$$
$$= \text{a rt. } \angle;$$

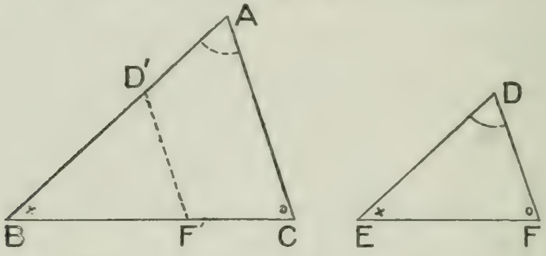
and, hence, a circle described on ST as diameter passes through P.

19. If L, M, N be three points in a st. line, and P a point at which LM and MN subtend equal $\angle$s, the locus of P is a circle.

Similar Triangles

Theorem 8

If the angles of one triangle are respectively equal to the angles of another, the corresponding sides of the triangles are proportional.



Hypothesis.—In $\triangle$s ABC, DEF; $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F.

To prove $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Proof.—Apply $\triangle$ DEF to $\triangle$ ABC so that $\angle$ E coincides with $\angle$ B; the $\triangle$ DEF taking the position D′BF′.

$\therefore \angle BD'F' = \angle A$, $\therefore D'F' \parallel AC$. (I—7, p. 38.)

$\therefore \frac{AB}{D'B} = \frac{CB}{F'B}$ (IV—2, Cor., p. 223.)

$\therefore \frac{AB}{DE} = \frac{BC}{EF}$.

In the same way, by applying the $\triangle$s so that $\angle$s C and F coincide, it may be proved that $\frac{BC}{EF} = \frac{CA}{FD}$.

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

Note.—$\because \frac{AB}{DE} = \frac{BC}{EF}$, $\therefore \frac{AB}{BC} = \frac{DE}{EF}$,

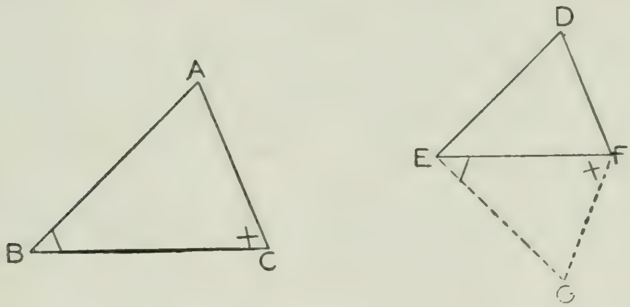
and in the same way $\frac{BC}{CA} = \frac{EF}{FD}$ and $\frac{CA}{AB} = \frac{FD}{DE}$.

$\therefore$ **If two triangles are similar, the corresponding sides about the equal angles are proportional.**

Theorem 9

(Converse of Theorem 8)

If the sides of one triangle are proportional to the sides of another, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle$s ABC, DEF; $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

To prove $\angle A = \angle D$, $\angle B = \angle DEF$, $\angle C = \angle DFE$.

Construction.—Make $\angle FEG = \angle B$, $\angle EFG = \angle C$.

Proof.—In $\triangle$s ABC, GEF $\begin{cases} \angle A = \angle G, \\ \angle B = \angle GEF, \\ \angle C = \angle EFG. \end{cases}$

$\therefore \quad \triangle ABC \parallel\!\mid \triangle GEF.$

$\therefore \ \frac{AB}{GE} = \frac{BC}{EF}.$ (IV—8, p. 236.)

But, by hypothesis, $\frac{AB}{DE} = \frac{BC}{EF}.$

$\therefore \ \frac{AB}{GE} = \frac{AB}{DE}, \quad \therefore \ GE = DE.$

Similarly it may be proved that $GF = DF$.

In $\triangle$s DEF, GEF $\begin{cases} DE = GE, \\ EF \text{ is common,} \\ FD = FG. \end{cases}$

$\therefore \ \triangle DEF \equiv \triangle GEF.$ (I—4, p. 22.)

$\therefore \ \angle DEF = \angle GEF = \angle B, \ \angle DFE = \angle GFE = \angle C.$

$\therefore$ remaining $\angle D$ = remaining $\angle A$.

126.—Exercises

1. The st. line joining the middle points of the sides of a △ is ‖ to the base, and equal to half of it.

2. If two sides of a quadrilateral be ‖, the diagonals cut each other proportionally.

3. In the △ **ABC** the medians **BE, CF** cut at **G.** Show that **BG** = twice **GE,** and **CG** = twice **GF.**

4. Using the theorem in Ex. 3, devise a method of trisecting a st. line.

5. If three st. lines meet at a point, they intercept on any ‖ st. lines portions which are proportional to one another.

6. In similar △s ⊥s from corresponding vertices to the opposite sides are in the same ratio as the corresponding sides.

7. In similar △s the bisectors of two corresponding ∠s, terminated by the opposite sides, are in the same ratio as the corresponding sides.

8. **ABCD** is a ‖gm, and a line through **A** cuts **BD** at **E, BC** at **F** and meets **DC** produced at **G.** Show that **AE : EF = AG : AF.**

9. If two ‖ st. lines **AB, CD** be divided at **E, F** respectively so that **AE : EB = CF : FD,** then **AC, BD** and **EF** are concurrent.

10. The median drawn to a side of a △ bisects all st. lines ‖ to that side and terminated by the other two sides, or those sides produced.

11. **ABCD** is a ‖gm. **AD** is bisected at **E** and **BC** at **F.** Show that **AF** and **CE** trisect the diagonal **BD.**

12. If the st. lines **OAB, OCD, OEF** be similarly divided, the △s **ACE, BDF** are similar.

13. If the corresponding sides of two similar △s be ||, the st. lines joining the corresponding vertices are concurrent.

14. △ **LMN** ||| △ **PQR**, ∠ **L** = ∠ **P** and ∠ **M** = ∠ **Q**. **LM** = 7 cm., **MN** = 5 cm., **LN** = 9 cm., **QR** = 4 cm. Find **PQ** and **PR**.

15. In △ **DEF**, **DE** = 13 cm., **EF** = 5 cm. and **DF** = 12 cm. The △ is folded so that the point **D** falls on the point **E**. Find the length of the crease.

16. **LMN** is a △ and **X** is any point in **MN**. Prove that the radii of the circles circumscribing **LMX**, **LNX** are proportional to **LM**, **LN**.

17. St. lines **POQ**, **ROS** are drawn so that **PO** = 2 **OQ** and **RO** = 2 **OS**. **RQ** and **PS** are produced to meet at **T**. Prove that **PS** = **ST** and **RQ** = **QT**.

18. **FDE**, **GDE** are two circles and **FDG** is a st. line. **FE**, **GE** are drawn. Prove that **FE** is to **GE** as diameter of circle **FDE** is to diameter of **GDE**.

19. **P** is any point on either arm of an ∠ **XOY**, and **PN** ⊥ to the other arm. Show that $\frac{PN}{OP}$ has the same value for all positions of **P**.

Show also that $\frac{ON}{OP}$ has the same value for all positions of **P**; and that $\frac{PN}{ON}$ has the same value for all positions of **P**.

(NOTE.—*The ratio* $\frac{PN}{OP}$ *is called the* **sine** *of the* ∠ **XOY**, $\frac{ON}{OP}$ *is the* **cosine** *of that* ∠, *and* $\frac{PN}{ON}$ *is the* **tangent** *of the same* ∠.)

20. PQRS is a quadrilateral inscribed in a circle. The diagonals PR, QS cut at X. Prove that $\frac{PQ}{SR} = \frac{XP}{XS}$.

21. OX, OY, OZ are three fixed st. lines, and P is any point in OZ. From P, PL is drawn $\perp$ OX and PM $\perp$ OY. Prove that the ratio PL : PM is constant.

22. In the quadrilateral DEFG the side DE $\parallel$ GF and the diagonals DF, EG cut at H. Through H the line LHM is drawn $\parallel$ DE and meeting EF, DG at L, M respectively. Prove HL = HM.

23. KLMN is a quadrilateral in which KL $\parallel$ NM. Prove that the line joining the middle points of KL and MN passes through the intersection of the diagonals KM, LN.

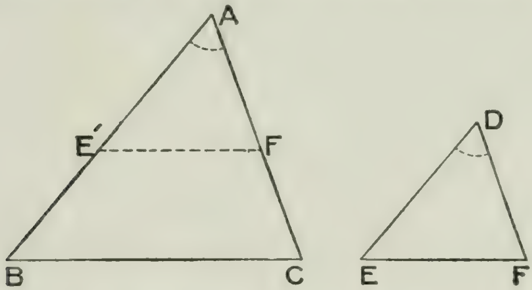
24. DEF is a $\triangle$ and G is any point in EF. The bisector of $\angle$ DGF meets DF in H. EH cuts DG at K. FK meets DE at L. Prove that LG bisects $\angle$ DGE.

25. DG and DH bisect the interior and exterior $\angle$s at D of a $\triangle$ DEF, and meet EF at G and H; and O is the middle point of EF. Show that OE is a mean proportional between OG and OH.

26. DG bisects $\angle$ D of $\triangle$ DEF and meets EF at G. GK bisects $\angle$ DGE and meets DE at K. GH bisects $\angle$ DGF and meets DF at H. Prove that $\triangle$ EKH : $\triangle$ FKH = ED : DF.

THEOREM 10

If two triangles have one angle of one equal to one angle of the other and the sides about these angles proportional, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle$s ABC, DEF, $\angle$ A = $\angle$ D

$$\text{and } \frac{AB}{DE} = \frac{AC}{DF}.$$

To prove $\triangle ABC \parallel\!\mid \triangle DEF$.

Proof.—Apply the $\triangle$s so that $\angle$ D coincides with $\angle$ A and $\triangle$ DEF takes the position AE′F′.

$$\because \frac{AB}{DE} = \frac{AC}{DF},$$

$$\therefore \frac{AB}{AE'} = \frac{AC}{AF'}, \quad \therefore E'F' \parallel BC. \quad \text{(IV—3, p. 224.)}$$

$$\therefore \angle B = \angle AE'F', \ \angle C = \angle AF'E', \quad \text{(I—9, p. 42.)}$$

$$\therefore \triangle ABC \parallel\!\mid \triangle AE'F'.$$

But $\triangle$ AE′F′ is the triangle DEF in its new position,

$$\therefore \triangle ABC \parallel\!\mid \triangle DEF.$$

The equal $\angle$s B, E are respectively opposite the corresponding sides AC, DF, also the equal $\angle$s C, F are respectively opposite the corresponding sides AB, DE.

Theorem 11

If two triangles have two sides of one proportional to two sides of the other, and the angles opposite one pair of corresponding sides in the proportion equal, the angles opposite the other pair of corresponding sides in the proportion are either equal or supplementary.

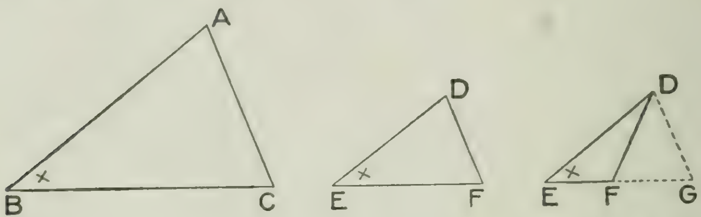


Fig. 1 Fig. 2

Hypothesis.—In $\triangle$s ABC, DEF, $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle B = \angle E$.

To prove either $\angle C = \angle F$ or $\angle C + \angle DFE = 2$ rt. $\angle$s.

Proof.—(1) If $\angle A = \angle D$. (Fig. 1.)

$\because \angle A = \angle D$, and $\angle B = \angle E$, $\therefore \angle C = \angle F$.

In this case $\triangle ABC \; ||| \; \triangle DEF$.

(2) If $\angle A$ is not equal to $\angle D$. (Fig. 2.)

At D make $\angle EDG = \angle A$ and produce DG to meet EF, produced if necessary, at G.

In $\triangle$s ABC, DEG $\left\{ \begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ \therefore \angle C = \angle G. \end{array} \right.$

$$\therefore \triangle ABC \; ||| \; \triangle DEG.$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DG}. \qquad \text{(IV—8, p. 236.)}$$

But, by hypothesis, $\frac{AB}{DE} = \frac{AC}{DF}$.

$$\therefore \frac{AC}{DG} = \frac{AC}{DF}, \quad \therefore \ DG = DF.$$

In $\triangle$ DFG, $\because$ DF = DG, $\therefore$ $\angle$ DGF = $\angle$ DFG.

But $\angle$ DGF = $\angle$ C, $\therefore$ $\angle$ DFG = $\angle$ C.

$\angle$ DFE + $\angle$ DFG = 2 rt. $\angle$s,

$\therefore$ $\angle$ DFE + $\angle$ C = 2 rt. $\angle$s.

127.—Exercises

1. Show that certain propositions of Book I are respectively particular cases of Theorems 9, 10 and 11 of Book IV.

2. In similar $\triangle$s medians drawn from corresponding vertices are proportional to the corresponding sides.

3. In a $\triangle$ ABC, AD is drawn $\perp$ BC. If BD : DA = DA : DC, prove that BAC is a rt. $\angle$.

4. If the diagonals of a quadrilateral divide each other proportionally, one pair of sides are $\parallel$.

5. A point D is taken within a $\triangle$ LMN and joined to L and M. A $\triangle$ EMN is described on the other side of MN from $\triangle$ LMN having $\angle$ EMN = $\angle$ DML, and $\angle$ ENM = $\angle$ DLM. Prove that $\triangle$ DME ||| $\triangle$ LMN.

6. M, N are fixed points on the circumference of a given circle, and P is any other point on the circumference. MP is produced to Q so that PQ : PN is a fixed ratio. Find the locus of Q.

7. EOD, GOF are two st. lines such that GO : DO = EO : FO. Prove that E, F, D, G are concyclic.

8. OEF, OGD are two st. lines such that OE : OG = OD : OF. Prove that E, F, G, D are concyclic.

9. DEF is a $\triangle$, and FX $\perp$ DE. Prove that, if DF : FX = DE : EF, $\angle$ XFE = $\angle$ D.

10. Similar isosceles $\triangle$s DEF, DEG are described on opposite sides of DE such that DF = DE and GD = GE. H is any point in DF and K is taken in GD such that GK : GD = DH : DF. Prove $\triangle$ KHE ||| $\triangle$ GDE.

11. LMN is a $\triangle$, and D is any point in LM produced. E is taken in NM such that NE : EM = LD : DM. Prove that DE produced bisects LN.

12. O is the centre and OD a radius of a circle. E is any point in OD, and F is taken in OD produced such that OF is a third proportional to OE, OD. P is any point on the circumference. Prove $\angle$ FPD = $\angle$ DPE.

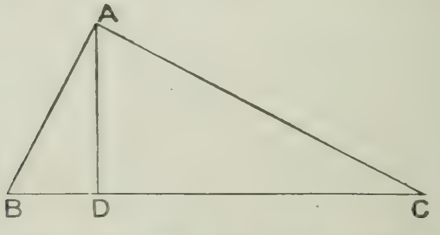
13. The bisectors of the interior and exterior $\angle$s at L in the $\triangle$ LMN meet MN and MN produced at D, E respectively. FNG drawn || LM meets LE at F and LD produced at G. Prove FN = NG.

14. If one pair of $\angle$s of two $\triangle$s be equal and another pair of $\angle$s be supplementary, the ratios of the sides opposite to these pairs of $\angle$s are equal to each other.

Geometric Means

Theorem 12

The perpendicular from the right angle to the hypotenuse in a right-angled triangle divides the triangle into two triangles which are similar to each other and to the original triangle.



Hypothesis.—In $\triangle$ ABC, $\angle$ BAC is a rt. $\angle$ and AD $\perp$ BC.

To prove $\triangle$ ABD ||| $\triangle$ CAD ||| $\triangle$ CBA.

Proof.—

In $\triangle$s ABD, CBA $\begin{cases} \angle \text{B is common.} \\ \angle \text{BDA} = \angle \text{BAC, both rt.} \angle\text{s.} \\ \therefore \angle \text{BAD} = \angle \text{BCA.} \end{cases}$

$\therefore$ $\triangle$ ABD ||| $\triangle$ CBA

Similarly $\triangle$ ADC ||| $\triangle$ CBA.

$\therefore$ $\triangle$ ABD ||| $\triangle$ CAD ||| $\triangle$ CBA.

Cor. 1.— $\because$ $\triangle$ ABD ||| $\triangle$ CAD,

$$\therefore \quad \frac{\text{BD}}{\text{AD}} = \frac{\text{AD}}{\text{CD}}.$$

$\therefore$ AD is the mean proportional between BD and DC.

Cor. 2.—Because $\triangle$ ABD ||| $\triangle$ CBA

$$\therefore \quad \frac{\text{BD}}{\text{AB}} = \frac{\text{AB}}{\text{BC}}.$$

$\therefore$ AB is the mean proportional between BD and BC.

Similarly—

AC is the mean proportional between DC and CB.

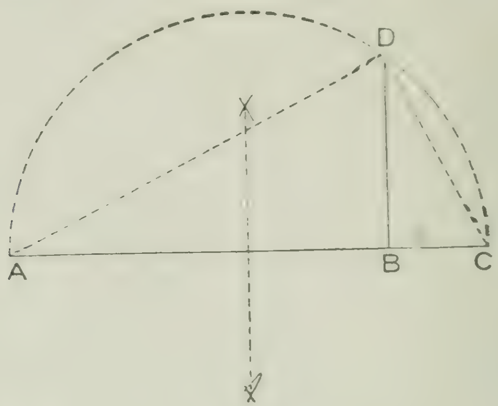
Cor. 3.—Because $\triangle$ CBA ||| $\triangle$ CAD,

$$\therefore \quad \frac{\text{CB}}{\text{BA}} = \frac{\text{CA}}{\text{AD}}.$$

i.e., the hypotenuse is to one side as the other side is to the perpendicular.

Problem 5

To find the mean proportional between two given straight lines.



From a st. line cut off AB, BC respectively equal to the two given st. lines.

It is required to find the mean proportional to AB, BC.

On AC as diameter describe a semi-circle ADC. From B draw BD $\perp$ AC and meeting the arc ADC at D.

BD is the required mean proportional.

Join AD, DC.

Proof.— $\because$ ADC is a semi-circle,

$\therefore$ $\angle$ ADC is a rt. $\angle$. (III—9, p. 160.)

In $\triangle$ ADC, $\angle$ ADC is a rt. $\angle$,
and DB $\perp$ AC.

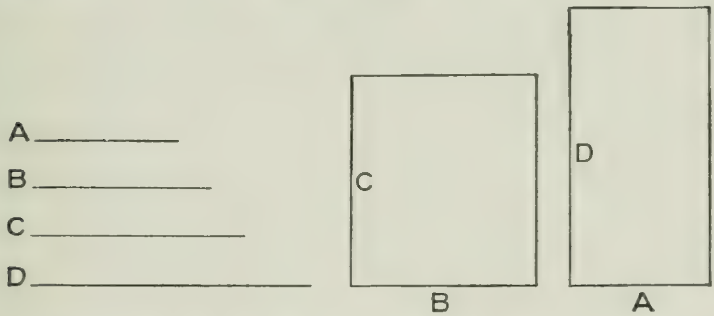
$\therefore \dfrac{AB}{BD} = \dfrac{DB}{BC}$. (IV—12, Cor. 1, p. 245.)

$\therefore$ BD is the mean proportional between AB and BC.

Rectangles

Theorem 13

If four straight lines are proportionals, the rectangle contained by the means is equal to the rectangle contained by the extremes.



Hypothesis.—A, B, C, D are four st. lines such that $\frac{A}{B} = \frac{C}{D}$.

To prove that rect. B.C = rect. A.D.

Let a, b, c, d be the numerical measures of A, B, C, D respectively.

$$\text{Then} \quad \frac{a}{b} = \frac{c}{d}.$$

$$\therefore bc = ad.$$

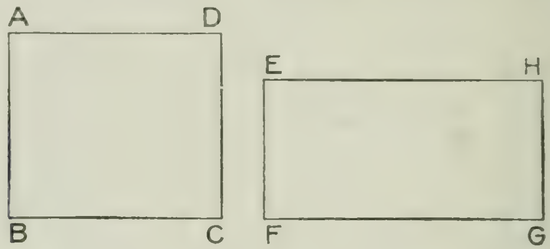
But bc is the numerical measure of B.C and ad is the numerical measure of A.D,

$$\therefore \text{rect. B.C} = \text{rect. A.D.}$$

Theorem 14

(Converse of Theorem 13)

If two rectangles are equal to each other, the length of one is to the length of the other as the breadth of the second is to the breadth of the first.



Hypothesis.—Rect. ABCD = rect. EFGH.

To prove $\frac{BC}{FG} = \frac{EF}{AB}$.

Proof.—Let a, b, c, d be the numerical measures of BC, BA, FG, EF respectively.

Then since the rectangles are equal,

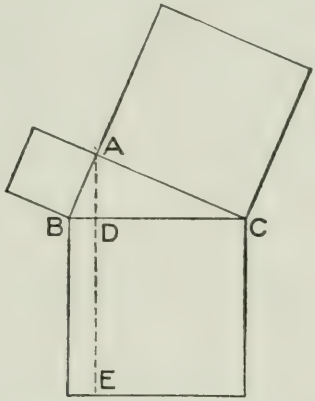
$$ab = cd.$$

$$\therefore \frac{a}{c} = \frac{d}{b},$$

$$\therefore \frac{BC}{FG} = \frac{EF}{AB}.$$

Alternative proof of the Pythagorean Theorem. *(II—13, p. 122.)*

The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.



Hypothesis.—BAC is a $\triangle$ having $\angle$ BAC a rt. $\angle$, and having squares described on the three sides.

To prove that $BC^2 = BA^2 + AC^2$.

Construction.—Draw $AD \perp BC$.

Proof.—$\because$ BAC is a rt.-$\angle$d $\triangle$ with $AD \perp$ the hypotenuse BC,

$$\therefore \quad \frac{BC}{BA} = \frac{BA}{BD}. \quad \text{(IV—12, Cor. 2, p. 245.)}$$

$$\therefore \quad BA^2 = BC.BD. \quad \text{(IV—13, p. 247.)}$$

$$\text{Similarly} \quad CA^2 = BC.CD.$$

$$\begin{aligned} \therefore \quad BA^2 + CA^2 &= BC.BD + BC.CD \\ &= BC\,(BD + CD) \\ &= BC.BC \\ &= BC^2 \end{aligned}$$

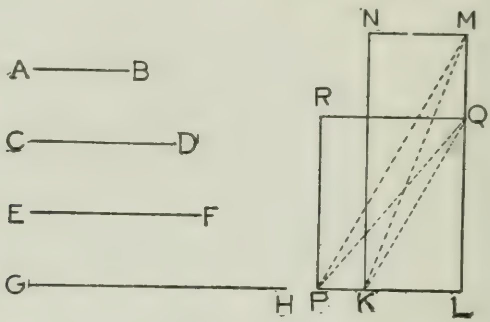
$$\textit{i.e.,}\ BC^2 = BA^2 + CA^2.$$

128.—Exercises

1. Give a general enunciation of IV—12, Cor. 1.

2. Give a general enunciation of IV—12, Cor. 2.

3. Give an alternative proof of IV—13, using the construction indicated in the following diagram:—



$\frac{AB}{CD} = \frac{EF}{GH}$. In the rectangles NL, RL, KL = AB, LM = GH, PL = CD and LQ = EF.

Using a similar construction give also an alternative proof of IV—14.

4. In any two equal $\triangle$s ABC, DEF, if AG, DH be $\perp$s to BC, EF respectively, AG : DH = EF : BC.

5. In any $\triangle$ the $\perp$s from the vertices to the opposite sides are inversely as the sides.

6. In the diagram of IV—12, show that rect. AD.BC = rect. BA.AC. Give a general statement of this theorem.

7. ABC, DEF are two equal $\triangle$s having also $\angle B = \angle E$. Show that $\frac{BC}{EF} = \frac{DE}{AB}$.

8. ABCD, EFGH are two equal $\|$gms having also $\angle B = \angle F$. Show that $\frac{BC}{FG} = \frac{FE}{BA}$.

9. ABCD is a given rect. and EF a given st. line. It is required to make a rect. equal in area to ABCD and having one of its sides equal to EF.

10. Make a rect. equal in area to a given △ and having one of its sides equal to a given st. line.

11. Show how to construct a rect. equal in area to a given polygon and having one of its sides equal to a given st. line.

12. If from any point on the circumference of a circle a ⊥ be drawn to a diameter, the square on the ⊥ equals the rect. contained by the segments of the diameter.

13. **Construct a square equal to a given rect.**

14. Construct a square equal to a given ||gm.

15. Construct a square equal to a given △.

16. Draw a square having its area 12 sq. inches.

17. Divide a given st. line into two parts such that the rect. contained by the parts is equal to the square on another given st. line.

18. If a st. line be divided into two parts, the rect. contained by the parts is greatest when the line is bisected.

19. AB and C are two given st. lines. Find a point D in AB produced such that rect. AD.DB = sq. on C.

20. Construct a rect. equal in area to a given square and having its perimeter equal to a given st. line.

When will the solution be impossible?

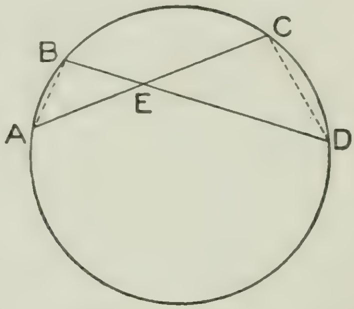
21. Show how to construct a square equal in area to a given polygon.

22. In the corresponding sides BC, EF of the similar △s ABC, DEF the points G, H are taken such that BG : GC = EH : HF. Prove AG : DH = BC : EF.

Chords and Tangents

Theorem 15

If two chords intersect within a circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.



Hypothesis.—In the circle ABC, the chords AC, BD intersect at E.

To prove that rect. AE.EC = rect. BE.ED.

Construction.—Join AB, CD.

Proof.— $\because$ $\angle$s ABD, ACD are in the same segment,

$\therefore$ $\angle$ ABD = $\angle$ ACD. (III—7, p. 156.)

Similarly, $\angle$ BAC = $\angle$ BDC.

And $\angle$ AEB = $\angle$ CED. (I—1, p. 13.)

$\therefore$ $\triangle$ AEB ||| $\triangle$ DCE.

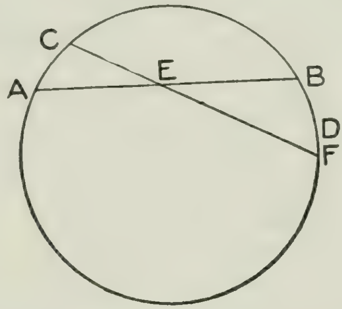
$\therefore$ $\dfrac{AE}{ED} = \dfrac{BE}{EC}$. (IV—8, p. 236.)

$\therefore$ rect. AE.EC = rect. BE.ED. (IV—13, p. 247.)

THEOREM 16

(Converse of IV—15)

If two straight lines cut each other so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, the four extremities of the two straight lines are concyclic.



Hypothesis—The st. lines AB, CD cut at E so that rect. AE.EB = rect. CE.ED.

To prove that A, C, B, D are concyclic.

Construction.—Describe a circle through A, C, B, and let it cut ED, produced if necessary, at F.

Proof.— ∵ AB, CF are chords of a circle,

∴ AE.EB = CE.EF. (IV—15, p. 252.)

But, AE.EB = CE.ED. (*Hyp.*)

∴ CE.EF = CE.ED.

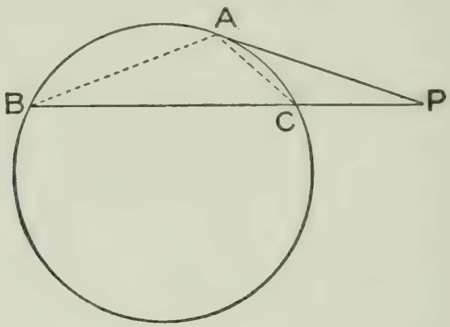
And ∴ EF = ED.

∴ F coincides with D,

and the points A, C, B, D are concyclic.

Theorem 17

If from a point without a circle, a secant and a tangent are drawn, the square on the tangent is equal to the rectangle contained by the secant, and the part of it without the circle.



Hypothesis.—PA is a tangent and PCB a secant to the circle ABC.

To prove that $PA^2 = PB.PC$.

Construction.—Join AB, AC.

Proof.— $\because$ AP is a tangent, and AC is a chord from the same point A,

$\therefore \angle PAC = \angle ABC$. (III—15, p. 177.)

In $\triangle$s PAB, PCA, $\begin{cases} \angle P \text{ is common,} \\ \angle PBA = \angle PAC, \\ \text{and } \therefore, \angle PAB = \angle PCA, \end{cases}$

$\therefore \triangle PAB \;|||\; \triangle PCA$.

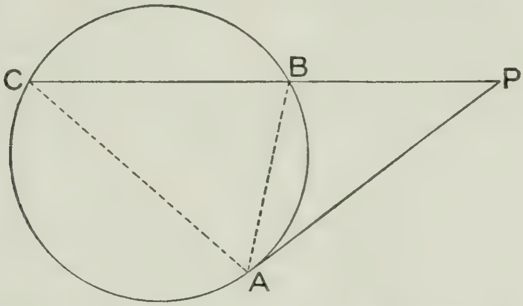
$\therefore \dfrac{PB}{PA} = \dfrac{PA}{PC}$. (IV—8, p. 236.)

$\therefore PA^2 = PB.PC$. (IV—13, p. 247.)

Theorem 18

(Converse of IV—17)

If from a point without a circle two straight lines are drawn, one of which is a secant and the other meets the circle so that the square on the line which meets the circle is equal to the rectangle contained by the secant and the part of it without the circle, the line which meets the circle is a tangent.



Hypothesis.—PA and PBC are drawn to the circle ABC so that $PA^2 = PB.PC$.

To prove that PA is a tangent.

Construction.—Join AB, AC.

Proof.—In △s PAB, PAC, ∠ P is common,

and ∵ $PA^2 = PB.PC$,

∴ $\dfrac{PA}{PB} = \dfrac{PC}{PA}$. (IV—14, p. 248.)

∴ △ PAB ||| △ PCA. (IV—10, p. 241.)

∴ ∠ PAB = ∠ PCA.

∴ PA coincides with the tangent at A. (III—15, p. 177.)

i.e., PA is a tangent to the circle.

Note.—*Prove this proposition with the following construction:—Draw a tangent from* P, *and join the point of contact and the points* A, P *to the centre.*

129.—Exercises

1. PAB, PCD are two secants drawn from a point P without a circle. Show that rect. PA.PB = rect. PC.PD.

From this exercise deduce a proof for IV—17.

2. If in two st. lines PB, PD points A, C respectively be taken such that rect. PA.PB = rect. PC.PD, the four points A, B, C, D are concyclic.

3. If two circles intersect, their common chord bisects their common tangents.

4. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal to each other.

5. Through P any point in the common chord, or the common chord produced, of two intersecting circles two lines are drawn cutting one circle at A, B and the other at C, D. Show that A, B, C, D are concyclic.

6. Through a point P within a circle, any chord APB is drawn. If O be the centre, show that rect. AP.PB = $OA^2 - OP^2$.

7. From any point P without a circle any secant PAB is drawn. If O be the centre, show that rect. PA.PB = $OP^2 - OA^2$.

8. From a given point as centre describe a circle cutting a given st. line in two points, so that the rectangle contained by their distances from a given point in the st. line may be equal to a given square.

9. Describe a circle to pass through two given points and touch a given st. line.

10. If three circles be drawn so that each intersects the other two, the common chords of each pair meet at a point.

11. Find a point D, in the side BC of $\triangle$ ABC, such that the sq. on AD = rect. BD.DC. When is the solution possible?

12. Use IV—17 to find a mean proportional to two given st. lines.

13. P is a point at a distance of 7 cm. from the centre of a circle. PDE is a secant such that PD = 5 cm. and DE = 3 cm. Find the length of the radius of the circle.

14. In a circle of radius 4 cm. a chord DE is drawn 7 cm. in length. F is a point in DE such that DF = 5 cm. Find the distance of F from the centre of the circle.

15. DEF is an isosceles $\triangle$ in which ED = EF. A circle, which passes through D and touches EF at its middle point cuts DE at H. Prove that DH = 3 HE.

16. In a circle two chords DE, FG cut at H. Prove that

$$(FH - HG)^2 - (DH - HE)^2 = FG^2 - DE^2.$$

17. LND, MNE are two chords intersecting inside a circle and LM is a diameter. Prove that

$$LN.LD + MN.ME = LM^2.$$

18. DEF, HGF are two circles and DFG is a fixed st. line. Show how to draw a st. line EFH such that EF.FH = DF.FG.

19. P is a point in the diameter DE of a circle, and PT is the $\perp$ on the tangent at a point Q. Prove that PT.DE = DP.PE + PQ^2.

20. P, Q, R, S are four points in order in the same st. line. Find a point O in this st. line such that OP.OR = OQ.OS.

21. The tangent at P to a circle, whose centre is O meets two $\parallel$ tangents in Q, R. Prove that PQ.PR = OP^2.

Miscellaneous Exercises

1. **EFGH** is a ||gm, **P** a point in **EF** such that **EP** : **PF** = $m:n$. What fraction is △ **EPH** of the ||gm?

2. **EFGH** is a ||gm, **P** is a point in the diagonal **FH** such that **FP** : **PH** = 2 : 5. What fraction of the ||gm is △ **EFP**? If **FP** : **PH** = $m:n$ find the fraction.

3. **EFGH** is a ||gm, **P** is a point in the diagonal **FH** produced such that **FP** : **PH** = 9 : 5. What fraction of the ||gm is the △ **PEH**?

4. **KLMN** is a ||gm. Any st. line **EKG** is drawn cutting the sides **ML** and **MN** produced at **E** and **G**. Show that half the ||gm is a mean proportional between △s **EKL** and **NKG**.

5. The △ **PQR** has **PQ** and **QR** divided at **D** and **E** such that **PD** : **DQ** = **QE** : **ER** = 1 : 3. **PE** and **RD** intersect at **O**. Find the ratios of the △s **PDO** : **OPR** : **OER** : **PQR**.

6. **D** and **E** are points in **PQ** and **PR** sides of the △ **PQR** such that **QD** : **DP** = **PE** : **ER** = $m:n$. Compare the areas of the △s **QDE** and **DER**.

7. Either of the complements of the ||gms about the diagonal of a ||gm is a mean proportional between the two ||gms about the diagonal.

8. **LMN** is an isosceles △ having **LM** = **LN**, **LD** is perpendicular to **MN**, **P** is a point in **LN** such that **LP** : **LM** = 1 : 3. Prove that **MP** bisects **LD**.

9. Through **E** one of the vertices of a rectangle **EFGH** any st. line is drawn, and **HP** and **FQ** are ⊥s to **PEQ**. Prove **PE.EQ** = **HP.FQ**.

10. **DEF** is a △, **P** and **Q** are points in **DE** and **DF**, and **DP** : **PE** = 3 : 5 and **DQ** : **QF** = 7 : 8. In what ratio is **PQ** cut by the median **DG**?

11. **DEFG** is a ||gm, and **EF** is produced to **K** so that **FK** = **EF**; **DK** cuts **EG** at **P**. Show that **GP** = $\frac{1}{3}$ **EG**.

12. The diagonals of the ||gm **EFGH** intersect at **O**; if **E** be joined to the middle point **P** of **OH**, and **EP** and **FG** meet at **K**, find **GK : EH**.

13. **DEF** is a right-angled △, **E** being the right angle. **G** is taken in **DE** produced such that **DG : GF** = **DF : EF**. Prove that ∠ **DFG** is right.

14. If the perpendicular to the base of a △ from the vertex be a mean proportional to the segments of the base, the triangle is right angled.

15. **DGH** is any △, and from **K** the middle point of **GH** a line is drawn cutting **DH** at **E** and **GD** produced at **F**. Prove **GF : FD** = **HE : ED**. Prove the converse also.

16. **AD** and **AE** are the interior and exterior bisectors of the vertical angle of △ **ABC** meeting the base at **D** and **E**. Through **C**, **FCG** is drawn || to **AB** meeting **AD** and **AE** at **F** and **G**. Prove that **FC** = **CG**.

17. **HKL** is an isosceles △, having **HK** = **HL**; **KL** is produced to **D** and **DEF** is drawn cutting **HL** at **E**, and **HK** at **F**. Prove **DE : DF** = **EL : KF**.

18. **DP** and **DQ** are perpendiculars to the bisectors of the interior angles **E** and **F** of any △ **DEF**. Prove **PQ** || **EF**.

19. **PX** and **QY** are perpendiculars from **P** and **Q** to **XY**; **PY** and **QX** intersect at **R**, and **RZ** is perpendicular to **XY**. Prove ∠ **PZX** = ∠ **QZY**.

20. **ABC** is any △, and **AD** is taken along **AC** such that **AC : AB** = **AB : AD**; also **CF** is taken along **AC** such that **AC : CB** = **CB : CF**. Prove **BF** = **BD**.

21. The perpendicular **KD** to the hypotenuse **HL** of a right-angled △ **KHL** is produced to **E** such that **KD : DH** = **DH : DE**. Prove **HE** || **KL**.

22. **DEF** is a $\triangle$ inscribed in a circle, and **P** and **Q** are taken in **DE** and **DF** such that **DP** : **PE** = **DQ** **QF**. Show that the circle described about **D**, **P**, **Q** touches the given circle at **D**.

23. **D** is a point in **LM** a side of $\triangle$ **LMN**, **DE** is $\parallel$ to **MN** and **EF** $\parallel$ to **LM**, meeting the sides at **E** and **F**. Prove **LD** : **DM** = **MF** : **FN**.

24. A variable line through a fixed point **O** meets two $\parallel$ st. lines at **P** and **Q**. Prove **OP** : **OQ** a constant ratio.

25. If the nonparallel sides of a trapezium are cut in the same ratio by a st. line, show that this line is $\parallel$ to the $\parallel$ sides.

26. **ABCDE** is a polygon, **O** a point within it. If **X**, **Y**, **Z**, **P**, **Q** are points in **OA**, **OB**, **OC**, **OD**, **OE** such that **OX** : **OA** = **OY** : **OB** = etc., show that the sides of **XYZPQ** are $\parallel$ to those of **ABCDE**.

27. **DE** is a st. line, **F** any point in it; find a point **P** in **DE** produced such that **PD** : **PE** = **DF** : **FE**.

28. St. lines **PD**, **PE**, **PF** and **PG** are such that each of the $\angle$s **DPE**, **EPF**, **FPG** is equal to half a right angle. **DEFG** cuts them such that **PD** = **PG**. Prove that **DG** : **FG** = **FG** : **EF**.

29. **GH** is a chord of a circle, **K** and **D** points on the two arcs respectively; **KH** and **KD** are joined and **GD** meets **KH** produced at **E**; **EF** $\parallel$ to **GH** meets **KD** produced. Show that **EF** is equal to the tangent from **F**.

30. **DEF**, **DEG** are two circles, the centre **P** of **DEG** being on the circumference of **DEF**. A st. line **PHGF** cuts the common chord at **H**. Prove that **PH** : **PG** = **PG** : **PF**.

31. **EF** is the diameter of a circle. **PQ** is a chord $\perp$ to **EF**, a chord **QXR** cuts **EF** at **X**, and **PR**, **EF** produced meet at **Y**. Show that **EX** : **EY** = **FX** : **FY**.

32. O is a fixed point and P a variable point on the circumference of a circle; PO is produced to Q such that OQ : OP = m n. Find the locus of Q.

33. LMN is a △ inscribed in a circle, ∠ L is bisected by LED cutting MN at E and the arc at D. Prove △s LEN and LMD similar.

34. The ∠ D of the △ DEF is bisected by DP cutting EF in P; QPR is ⊥ to DP meeting DE and DF at Q and R; RS is || to EF meeting DE at S. Prove SE = EQ.

35. AOB, COD and EOF are any three st. lines; ACE is || to FDB. Prove AC : CE = BD : DF. State and prove a converse to this theorem.

36. Two circles DEF and DEG intersect; a tangent DF is drawn to DEG, and EG to DEF. Show that DE is a mean proportional between FE and DG.

37. EFGH is a quadrilateral, the diagonals EG and FH meet at Q. Prove △ EFH : △ FGH = EQ : QG.

38. EFGH is a quadrilateral of which the sides EH and FG produced meet at P. Prove △ EFG : △ FGH = EP : PH.

39. G is the middle point of the st. line MN, PE a st. line || to MN. Any st. line EFGH cuts PN at F and PM produced at H. Prove EF : FG = EH : HG.

40. ABC is a △ having ∠ B = ∠ C = twice ∠ A, BD bisects the ∠ B meeting AC at D. Prove AC : AD = AD : DC; also prove △ ABC : △ ABD = △ ABD : △ BDC.

41. EFGH is a cyclic quadrilateral, EG and FH intersect at O, and OP and OQ are ⊥s to EH and FG. Show that OP : OQ = EH : FG.

42. EF is the diameter of a circle and P and Q any points on the circumference on opposite sides of EF; QR is ⊥ to EF meeting EP at S. Prove △ ESQ ||| △ EQP.

43. **ABC** is a △ inscribed in a circle, centre **O**, **AD** a ⊥ to **BC**, **AOE** a diameter. Prove △s **ADC** and **ABE** similar: and **AD.AE = AB.AC.**

44. **EFG** is a △ inscribed in a circle, **ED** ‖ to the tangent at **G** meets the base at **D**. Prove that **FG : FE = EG : ED.**

45. Find the ratio of the segments of the hypotenuse of a right-∠d △ made by a perpendicular on it from the vertex, if the ratio of the sides be (1) 1 : 2; (2) $m : n$.

46. **PQ** is the diameter of a circle; a tangent is drawn from a point **R** on the circumference, **PS** and **QT** are ⊥ to the tangent. Prove △s **PRQ, RPS** and **RTQ** similar; also show that △ **PRQ** is half of **PSTQ.**

47. **PQ** and **PR** are tangents to a circle, **PST** is a secant meeting the circle at **S** and **T**. Prove **QT : QS = RT : RS.**

48. Two circles intersect at **E** and **F**; from **P**, any point on one of them, chords **PED, PFG** are drawn, **EF** and **DG** meet at **Q** and **PQ** cuts the circle **PEF** at **R**. Prove **R, F, G, Q** concyclic; also that **PQ**^{2} is equal to the sum of the squares on the tangents to the circle **EFGD** from **P** and **Q.**

49. **PBR** is a st. line, and similar segments of circles, **PAB** and **BAR**, are described on **PB** and **BR** and on the same side of **PR**. **PAC** and **RAD** are drawn to meet the circles at **C** and **D**. Prove **PD : RC = PB : BR.**

NOTE.—*Segments of circles are said to be similar when they contain equal angles.*

50. **PMQ** is the diameter of a circle **PRQ**, **PX** and **QY** are ‖ tangents, **XRY** is any other tangent, **PY** and **XQ** meet at **O**. Show that **RO** is ‖ to **PX**; that **RO** produced to **M** is ⊥ to the diameter; and that **MO = OR.**

51. **ABCD** is a rectangle, a st. line **APQR** is drawn cutting **BC** at **P**, the circle circumscribing the rectangle at **Q** and **DC** produced at **R**, and such that **AC** bisects ∠ **DAR**. Prove **DC : CR = PQ : PA.**

52. **PQRS** is a square. A st. line **PFED** cuts **QS** at **F**, **SR** at **E** and **QR** produced at **D**. Prove **FR** a tangent to the circle described about **DER**; also that **EF** : **PF** = **PF** : **FD**.

53. **FGHK** is a cyclic quadrilateral, the ∠ **GFE** is made equal to ∠ **HFK** and **E** is in **GK**. Prove △s **FEK** and **FGH** similar.

54. **PA** and **PB** are tangents to a circle, centre **O**, **AB** meets **PO** in **R**; **PCD** is any secant, **OS** is ⊥ to **PD**, and **AB** and **OS** produced meet at **Q**. Prove (1) **P**, **R**, **S**, **Q** concyclic; (2) $PO.OR = OA^2$; (3) **QD** and **QC** are tangents to the given circle.

55. **DEF** is a △ and **P** and **Q** are points in **ED** and **FE** such that **EP** : **PD** = **FQ** : **QE**, and **PQ** meets **DF** produced at **R**. Prove $RF : RD = PE^2 : PD^2$. (*Through* **F** *draw a st. line* || *to* **DE** *to meet* **PR**.)

56. If a square is inscribed in a rt.-∠d △ having one side on the hypotenuse, show that the three segments of the base are in continued proportion.

57. **FGH** is a △ and ∠ **G** and ∠ **H** are bisected by st. lines which cut the opposite sides at **D** and **E**; if **DE** is || to **GH**, then **FG** = **FH**.

58. From **P**, the middle point of an arc of a circle cut off by a chord **QR**, any chord **PDE** is drawn cutting **QR** at **D**. Show that $PQ^2 = PD.PE$.

59. Draw a st. line through a given point so that the perpendiculars on it from two other given points may be (1) equal, (2) one twice the other, (3) three times the other, (4) in a given ratio.

60. **LMN** is an isosceles △, the base **MN** is produced both ways, in **NM** produced any point **P** is taken, and in **MN** produced **NQ** is taken a third proportional to **PM** and **LM**. Prove △s **PLQ** and **PLM** similar.

61. EDOF is the diameter of a circle, centre O. PE and PG are tangents to the circle; GD is ⊥ to EF. Prove GD : DE = OE : EP.

62. DEF is a △ inscribed in a circle, centre O. The diameter ⊥ to EF cuts DE at P and FD produced at Q. Prove △s EPO and FOQ similar; and hence OE^2 = OP.OQ.

63. ABC is a △ inscribed in a circle. The exterior ∠ at A is bisected by a st. line cutting BC produced at D and the circumference at E. Prove BA.AC = EA.AD.

64. EFGH is a cyclic quadrilateral, P a point on the circumference, PQ, PR, PS, PT are ⊥ to EF, FG, GH, HE respectively. Prove △s PTQ and PSR similar; and PT.PR = PS.PQ.

65. Any three || chords AB, CD, EF are drawn in a circle, AC and BD meet EF produced at Q and R, P is a point in the arc EF, and PA and PD meet EF at M and N. Prove △s AQM and NDR similar; hence show that, for all positions of P, QM.NR is constant.

66. Two tangents TMP and TNQ are drawn to a circle, centre O, and the st. line POQ is ⊥ to TO. MN is any other tangent to the circle. Prove △s MPO and NQO similar.

67. DH is a median of the △ DEF, PQ is || to EF cutting DE at P and DF at Q. Show that PF and EQ intersect on DH.

68. LNM is a △ inscribed in a semicircle, diameter LM. NM is greater than NL. On opposite sides of LN the ∠LNP is made equal to ∠LNQ, P and Q lying along LM. Prove PL : LQ = PM : QM.

69. EFGH is a ||gm, and RS is drawn || to HF meeting EH and EF at R and S. Show that RG and SG cut off equal segments of the diagonal FH. Prove a converse of this.

70. **ABC** is a △ and **AB, AC** are produced to **D, E** so that **BD** = **CE**; **DE** and **BC** produced meet at **F.** Show that **AD : AE** = **FC : FB.**

71. Two circles, centres **O, P** intersect, the centre **O** being on the circumference of the other circle. **GDE** touches the circle with centre **O** at **G** and cuts the other at **D, E,** and **EPF** is a diameter. Prove △ **OGD** ||| △ **OEF**; and hence, that **OD.OE** is constant for all positions of the tangent.

72. Two circles touch externally at **P**; **EF** a chord of one circle touches the other at **D.** Prove **PE : PF** = **ED : DF.**

73. **EOF** is the diameter of a circle, with centre **O, DP** any chord cutting the diameter; **OSQR** ⊥ to **DP** meets **DP** at **S, DE** at **Q,** and **PE** at **R.** Prove △s **EDF** and **RSP** similar; also **OQ.OR** = **OD**^{2}.

74. Divide an arc of a circle into two parts so that the chords which cut them off shall have a given ratio to each other.

75. **LMN** is a △, and **XY** ∥ **MN** meets **LM** at **X** and **LN** at **Y**; **MN** is produced to **D** so that **ND** = **XY,** and **XP** ∥ to **LD** meets **MN** at **P.** Prove **MN : ND** = **ND : NP.**

76. Two circles intersect and a st. line **CDOEF** cuts the circumferences at **C, D, E, F** and the common chord at **O.** Show that **CD : DO** = **EF : OE**

77. **DX** ⊥ **EF** and **EY** ⊥ **DF** in △ **DEF.** The lines **DX, EY** cut at **O.** Prove that **EX : XO** = **DX : XF.**

78. From a point **P** without a circle two secants **PKL, PMN** are drawn to meet the circle in **K, L, M, N.** The bisector of ∠ **KPM** meets the chord **KM** at **E** and the chord **LN** at **F.** Prove that **LF : FN** = **ME : EK**

79. **QR** is a chord ∥ to the tangent at **P** to a circle. A chord **PD** cuts **QR** at **E.** Prove that **PQ** is a mean proportional between **PE** and **PD.**

80. **DEF, DEG** are two fixed circles and **FEG** is a st. line. Show that the ratio **FD : DG** is constant for all positions of the st. line **FEG.**

81. **DEF** is a st. line, and **EG, FH** are any two ∥ st. lines on the same side of **DEF** such that **EG : FH = DE : DF.** Prove that **D, G, H** are in a st. line.

82. From a given point on the circumference of a circle draw two chords which are in a given ratio and contain a given ∠.

83. **DEF** is a △ and on **DE, DF** two △s **DLE, DFM** are described externally such that ∠ **FDM** = ∠ **EDL** and ∠ **DFM** = ∠ **DLE.** Prove △ **DLF** ||| △ **DEM.**

84. **DEFG** is a ∥gm and **P** is any point in the diagonal **EG.** The st. line **KPL** meets **DE** at **K** and **FG** at **L,** and **MPN** meets **EF** at **M** and **GD** at **N.** Prove **KM** ∥ **NL.**

85. **ABCD** is a ∥gm and **PQ** is a st. line ∥ **AB.** The st. lines **PA, QB** meet at **R** and **PD, QC** meet at **S.** Prove **RS** ∥ **AD.**

86. If the three sides of one △ are respectively ⊥ to the three sides of another △, the two △s are similar.

87. Find a point whose ⊥ distances from the three sides of a △ are in the ratio 1 : 2 : 3.

88. Squares are described each with one side on one given st. line and one vertex on another given st. line. Find the locus of the vertices which are on neither.

89. If the sides of a rt.-∠d △ are in the ratio 3 : 2, prove that the ⊥ from the vertex of the rt. ∠ to the hypotenuse divides it in the ratio 9 : 4.

90. **HK** is a diameter of a circle and **L** is any point on the circumference. A st. line ⊥ **HK** meets **HK** at **D, HL** at **E, KL** at **G,** and the circumference at **F.** Show that $DF^2 = DE.DG$.

91. The st. line joining a fixed point to any point on the circumference of a given circle is divided in a given ratio at **P**. Prove that the locus of **P** is a circle.

92. **DEFG** is a quadrilateral and **P, Q, R, S** are points on **DE, EF, FG, GD** such that **DP : DE = FQ : FE = FR : FG = DS : DG**. Prove that **PQRS** is a ||gm.

93. **DEFG** is a ||gm, and a line is drawn from **E** cutting **DF** in **P**, **DG** in **Q** and **FG** produced in **R**. Prove that $PQ : PR = DP^2 : PF^2$; and that $PQ.PR = EP^2$.

94. If $\triangle$ **DEF** : $\triangle$ **GHK = DE.EF : GH.HK**, prove that $\angle$s **E, H** are either equal or supplementary.

95. From a point **P** without a circle draw a secant **PQR**, such that **QR** is a mean proportional between **PQ** and **PR**.

96. Through a point of intersection of two circles draw a line such that the chords intercepted by the circles are in a given ratio.

97. If two $\triangle$s are on equal bases and between the same ||s, the intercepts made by the sides of the $\triangle$s on any st. line || to the base are equal.

98. The radius of a fixed circle is 38 mm., and a chord **LM** of the circle is divided at **P** such that **LP.PM** = 225 sq. mm. Construct the locus of **P**.

99. If the tangents from a given point to any number of intersecting circles are all equal, all the common chords of the circles pass through that point.

100. Circles are described passing through two fixed points; find the locus of a point from which the tangents to all the circles are equal.

101. **DEF** is a $\triangle$ having $\angle$ **E** a rt. $\angle$. A circle is described with centre **D** and radius **DE**; from **F** a secant is drawn cutting the circle at **G, H**; and **EX** is drawn $\perp$ **DF**. Show that **D, X, G, H** are concyclic.

102. **GD** is a chord drawn $\parallel$ to the diameter **LM** of a circle. **LG**, **LD** cut the tangent at **M** at **E**, **F** respectively. Prove that $LG.GE + LD.DF = LM^2$.

103. **LM** is a diameter of a circle, and on the tangent at **L** equal distances **LP**, **PQ** are cut off. **MP**, **MQ** cut the circumference at **R**, **S** respectively. Prove that $LR : RS = LM : MS$.

104. **GH** drawn in the $\triangle$ **DEF** meets **DE** in **G** and **DF** in **H**. From **D** any line **DLK** is drawn cutting **GH** in **L** and **EF** in **K**. From **L** the st. lines **LM**, **LN** are drawn $\parallel$ **KH**, **KG** and meeting **DH**, **DG** at **M**, **N** respectively. Prove $\triangle$ **LMN** $|||$ $\triangle$ **KHG**.

105. In a given $\triangle$ inscribe an equilateral $\triangle$ so as to have one side $\parallel$ to a side of the given $\triangle$.

106. In a given $\triangle$ **DEF** draw a st. line **PQ** $\parallel$ **ED** meeting **EF** in **P** and **DF** in **Q**, so that **PQ** is a mean proportional between **EP** and **PF**.

107. Two circles intersect at **E**, **F**, and **DEG** is the st. line $\perp$ **EF** and terminated in the circumferences. **HEK** is any other st. line through **E** terminated in the circumferences. **HF**, **DF**, **KF**, **GF** are drawn. Prove, by similar $\triangle$s, that $DG > HK$.

108. In $\triangle$ **ABC** the bisectors of $\angle$ **A** and of the exterior $\angle$ at **A** meet the st. line **BC** at **D** and **E**. Show that $DE = \dfrac{2\,abc}{c^2 - b^2}$.

109. If two circles intercept equal chords **PQ**, **RS** on any st. line, the tangents **PT**, **RT** to the circles at **P**, **R** are to one another as the diameters of the circles.

110. **DEF** is a $\triangle$ having $DF > DE$. From **DF** a part **DG** is cut off equal to **DE**, and **GH** is drawn $\parallel$ **DE** to meet

EF at H. From GF a part GK is cut off equal to GH, and KL is drawn || GH to meet EF at L; etc. Prove that DE, GH, KL, etc., are in continued proportion.

111. A circle P touches a circle Q internally, and also touches two || chords of Q. Prove that the ⊥ from the centre of P on the diameter of Q which bisects the chords is a mean proportional between the two extremes of the three segments into which the diameter is divided by the chords.

112. PX is the ⊥ from a point P on the circumference of a circle to a chord QR, and QY, RZ are ⊥s to the tangent at P. Prove that $PX^2 = QY.RZ$.

113. Prove, by using 112, that if ⊥s are drawn to the sides and diagonals of a cyclic quadrilateral from a point on the circumference of the circumscribed circle, the rectangle contained by the ⊥s on the diagonals is equal to the rectangle contained by the ⊥s on either pair of opposite sides.

114. The projections of two || st. lines on a given st. line are proportional to the st. lines.

115. DEFG is a square, and P is a point in GF such that DP = FP + FE. Prove that the st. line from D to the middle point of EF bisects ∠ PDE.

116. DEF, GEF are △s on opposite sides of EF, and DG cuts EF at H. Prove that △ DEF : △ GEF = DH : HG.

117. From the intersection of the diagonals of a cyclic quadrilateral ⊥s are drawn to a pair of opposite sides: prove that these ⊥s are in the same ratio as the sides to which they are drawn.

118. P, Q, R, S are points in a st. line, PX || QY, RX || SY, and XY meets PS at O. Prove that OP.OS = OQ.OR.

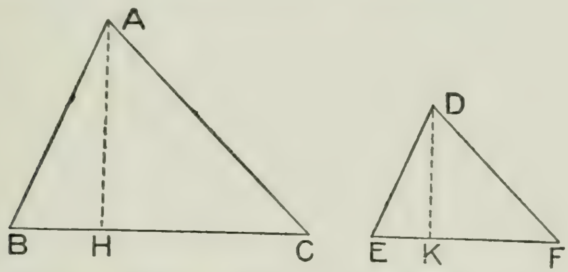
119. From a point T without a circle tangents TP, TQ and a secant TRS are drawn. Prove that in the quadrilateral PRQS the rect. PR.QS = the rect. RQ.SP.

BOOK V

Areas of Similar Figures

Theorem 1

The areas of similar triangles are proportional to the squares on corresponding sides.



Hypothesis.—ABC, DEF are similar $\triangle$s of which BC, EF are corresponding sides.

To prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{BC^2}{EF^2}$

Construction.—Draw $AH \perp BC$ and $DK \perp EF$.

Proof.—

$$\because \triangle AHC \;|||\; \triangle DKF,$$

$$\text{And } \triangle ABC \;|||\; \triangle DEF,$$

$$\therefore \frac{AH}{DK} = \frac{AC}{DF} = \frac{BC}{EF}. \quad \text{(IV—8, p. 236.)}$$

$$\triangle ABC = \tfrac{1}{2}\, AH.BC, \quad \text{(II—4, p. 100.)}$$

$$\triangle DEF = \tfrac{1}{2}\, DK.EF,$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\tfrac{1}{2}\, AH.BC}{\tfrac{1}{2}\, DK.EF}$$

$$= \frac{AH}{DK}.\frac{BC}{EF}$$

$$= \frac{BC}{EF}.\frac{BC}{EF}$$

$$= \frac{BC^2}{EF^2}.$$

130.—Exercises

1. Two similar $\triangle$s have corresponding sides in the ratio of 3 to 5. What is the ratio of their areas?

2. The ratio of the areas of two similar $\triangle$s equals the ratio of 64 to 169. What is the ratio of their corresponding sides?

3. Draw a $\triangle$ having sides 4 cm., 5 cm., 6 cm. Make a second $\triangle$ having its area four times that of the first, and divide it into parts each equal and similar to the first $\triangle$.

4. Show that the areas of similar $\triangle$s are as:—

   (*a*) the squares on corresponding altitudes;

   (*b*) the squares on corresponding medians;

   (*c*) the squares on the bisectors of corresponding $\angle$s.

5. **ABC, DEF** are two similar $\triangle$s such that area of $\triangle$ **DEF** is twice that of $\triangle$ **ABC**. What is the ratio of corresponding sides?

Draw $\triangle$ **ABC** having sides 5 cm., 6 cm., 7 cm., and make $\triangle$ **DEF** similar to $\triangle$ **ABC**, and of double the area.

6. If **ABC, DEF** be similar $\triangle$s of which **BC, EF** are corresponding sides, and the st. line **G** be such that **BC : EF = EF : G**, then $\triangle$ **ABC** : $\triangle$ **DEF = BC : G**; that is:—

If three st. lines be in continued proportion, the first is to the third as any $\triangle$ on the first is to the similar $\triangle$ similarly described on the second.

Note.—*Similar $\triangle$s are said to be similarly described on corresponding sides.*

7. **ABC** is a $\triangle$ and **G** is a st. line. Describe a $\triangle$ **DEF** similar to $\triangle$ **ABC** and such that $\triangle$ **ABC** : $\triangle$ **DEF = BC : G.**

Describe another $\triangle$ **HKL** similar to $\triangle$ **ABC** and such that $\triangle$ **ABC** : $\triangle$ **HKL = AB : G.**

8. Bisect a given △ by a st. line drawn ∥ to one of its sides.

9. From a given △ cut off a part equal to one-third of its area by a st. line drawn ∥ to one of its sides.

10. Trisect a given △ by st. lines drawn ∥ to one of its sides.

11. Show that the equilateral △ described on the hypotenuse of a rt.-∠d △ equals the sum of the equilateral △s on the two sides.

12. In △ DEF, DX ⊥ EF and EY ⊥ FD. Prove that △ FXY : △ DEF = $FX^2 : FD^2$.

13. In the acute-∠d △ DEF, DX ⊥ EF, EY ⊥ FD, FZ ⊥ DE, YG ⊥ EF and ZH ⊥ EF. Prove that XY and XZ divide the △ DEF into three parts that are proportional to FG, GH and HE.

14. LMN is an equilateral △. The st. lines RLQ, PMR, QNP are respectively ⊥ LM, MN, NL. Find the ratio of △ PQR to △ LMN.

15. A point O is taken in the diameter PQ produced of a circle. OT is a tangent, and the tangent at P cuts OT at N. If D is the centre of the circle, prove that △ OPN : △ OTD = OP : OQ.

16. H is a point on the circumference of a circle of which FG is a diameter, and O is the centre. HD ⊥ FG, and tangents at F and H meet at E. Prove that △ FEH : △ OHG = FD : DG.

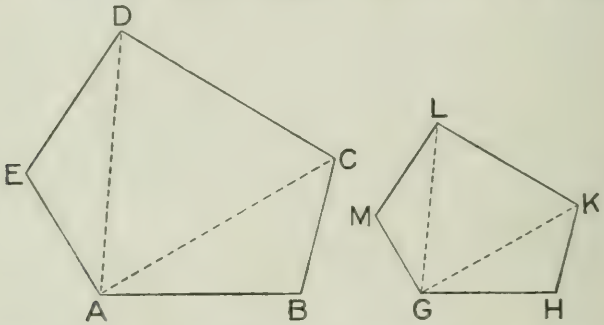
17. DEF, LMN are two △s in which ∠ E = ∠ M. Prove that △ DEF : △ LMN = DE.EF : LM.MN.

18. Similar △s are to one another as the squares on the radii of their circumscribing circles.

131. **Definition.**—If two polygons of the same number of sides have the angles of one taken in order around the figure respectively equal to the angles of the other in order, and have also the corresponding sides in proportion, the polygons are said to be **similar polygons.**

PROBLEM 1

To describe a polygon similar to a given polygon, and with the corresponding sides in a given ratio.



Let ABCDE be the given polygon, and GH a st. line taken such that AB is to GH in the given ratio.

It is required to describe on GH a polygon similar to ABCDE and such that AB and GH are corresponding sides.

Join AC, AD.

Make $\angle$ H = $\angle$ B, $\angle$ HGK = $\angle$ BAC and produce the arms to meet at K. Make $\angle$ KGL = $\angle$ CAD, $\angle$ GKL = $\angle$ ACD, and produce the arms to meet at L. Make $\angle$ LGM = $\angle$ DAE, $\angle$ GLM = $\angle$ ADE and produce the arms to meet at M.

GHKLM is the required polygon.

$\angle$ H = $\angle$ B, $\angle$ HGK = $\angle$ BAC, $\therefore$ $\angle$ HKG = $\angle$ BCA.

Similarly $\angle$ GLK = $\angle$ ADC, and $\angle$ M = $\angle$ E.

Hence $\angle$ HKL = $\angle$ BCD, $\angle$ KLM = $\angle$ CDE and $\angle$ HGM = $\angle$ BAE.

$\therefore$ polygon GHKLM has its $\angle$s equal respectively to the $\angle$s of polygon ABCDE.

From the similar $\triangle$s GHK, ABC, $\frac{GH}{AB} = \frac{HK}{BC} = \frac{KG}{CA}$;

and from the similar $\triangle$s GKL, ACD, $\frac{KG}{CA} = \frac{KL}{CD}$;

$$\therefore \frac{GH}{AB} = \frac{HK}{BC} = \frac{KL}{CD}.$$

In the same manner it may be shown that each of these ratios equals $\frac{LM}{DE}$ and $\therefore$ equals $\frac{MG}{EA}$.

Hence the corresponding sides of the two polygons are proportional; $\therefore$ polygon GHKLM is similar to polygon ABCDE; and the two polygons have their corresponding sides in the given ratio.

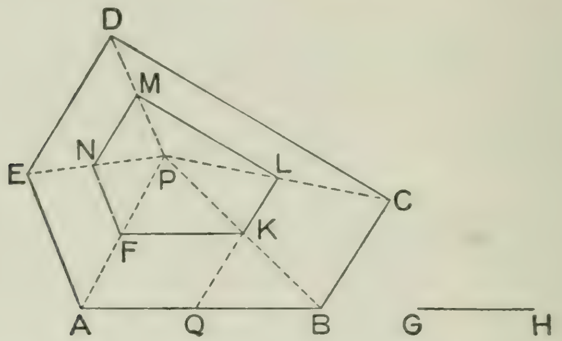
132.—Exercises

1. Draw diagrams to show that two quadrilaterals may have the sides of one respectively proportional to the sides of the other, but the $\angle$s of one not equal to the corresponding $\angle$s of the other.

2. Draw diagrams to show that two quadrilaterals may have the $\angle$s of one respectively equal to the $\angle$s of the other, but the corresponding sides not in the same proportion.

3. KLMN is a polygon. Construct a polygon similar to KLMN, and having each side one-third of the corresponding side of KLMN.

4. ABCDE is a given polygon and GH a given st. line. Cut off AQ = GH. Take any point P within ABCDE.



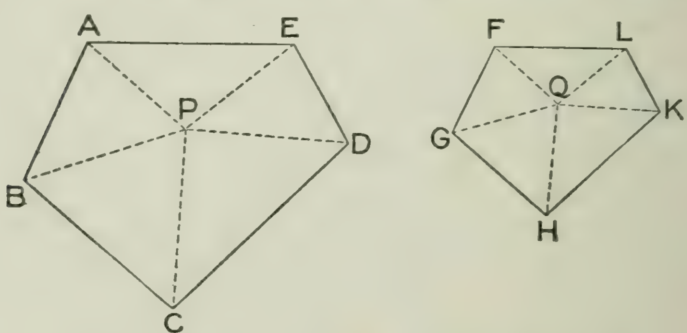
Join P to A, B, C, D, E. Draw QK ∥ AP, KF ∥ AB, FN ∥ AE, NM ∥ ED, KL ∥ BC. Join LM.

Show that FKLMN is similar to ABCDE.

5. Twice as many polygons may be described on a given st. line GH, each similar to a given polygon, as the given polygon has sides.

Problem 2

To divide similar polygons into similar triangles.



Let ABCDE, FGHKL be similar polygons of which AB and FG are corresponding sides.

It is required to divide ABCDE, and FGHKL into similar $\triangle$s.

Take any point P within the polygon ABCDE. Join PA, PB, PC, PD, PE.

Make $\angle$ GFQ = $\angle$ BAP and $\angle$ FGQ = $\angle$ ABP, and let the arms of these $\angle$s meet at Q.

Join QH, QK, QL.

$\angle$ PAB = $\angle$ QFG and $\angle$ PBA = $\angle$ QGF; $\therefore$ $\angle$ FQG = $\angle$ APB, and consequently $\triangle$s ABP, FGQ are similar;

$$\therefore \frac{QG}{PB} = \frac{FG}{AB}.$$

But, by definition of similar polygons,

$$\frac{FG}{AB} = \frac{GH}{BC}.$$

$$\therefore \frac{QG}{PB} = \frac{GH}{BC}.$$

Also $\angle$ FGH = $\angle$ ABC and $\angle$ FGQ = $\angle$ ABP;

$$\therefore \angle QGH = \angle PBC.$$

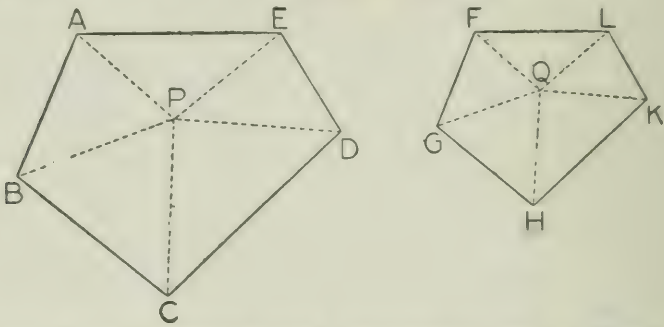
Then in $\triangle$s QGH, PBC, $\frac{QG}{PB} = \frac{GH}{BC}$, and $\angle$ QGH = $\angle$ PBC.

$\therefore$ these $\triangle$s are similar. (IV—10, p. 241.)

In the same manner it may be shown that the remaining pairs of corresponding $\triangle$s are similar.

Theorem 2

The areas of similar polygons are proportional to the squares on corresponding sides.



Using the diagram and construction of Problem 2.

It is required to show that $\dfrac{\text{polygon FGHKL}}{\text{polygon ABCDE}} = \dfrac{FG^2}{AB^2}$.

$\because$ $\triangle$s FGQ, ABP are similar,

$$\therefore \frac{\triangle FGQ}{\triangle ABP} = \frac{GQ^2}{BP^2}. \qquad \text{(V—1, p. 271.)}$$

$$\text{Similarly } \frac{\triangle QGH}{\triangle PBC} = \frac{GQ^2}{BP^2}.$$

$$\therefore \frac{\triangle QGF}{\triangle PAB} = \frac{\triangle QGH}{\triangle PBC} = \text{(in the same manner)}$$

$$\frac{\triangle QHK}{\triangle PCD} = \frac{\triangle QKL}{\triangle PDE} = \frac{\triangle QLF}{\triangle PEA}.$$

But, if any number of fractions be equal to each other, the sum of their numerators divided by the sum of their denominators equals each of the fractions.

Now the sum of the numerators of the equal fractions is the polygon FGHKL, and the sum of the denominators is the polygon ABCDE;

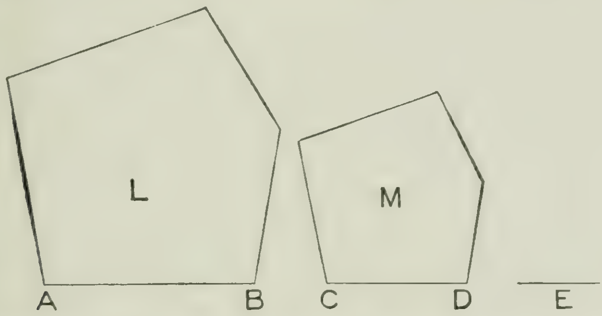
$$\therefore \frac{\text{polygon FGHKL}}{\text{polygon ABCDE}} = \frac{\triangle QFG}{\triangle PAB}.$$

$$\text{But } \frac{\triangle\ QFG}{\triangle\ PAB} = \frac{FG^2}{AB^2}.$$

$$\therefore \frac{\text{polygon } FGHKL}{\text{polygon } ABCDE} = \frac{FG^2}{AB^2}.$$

Theorem 3

If three straight lines are in continued proportion, the first is to the third as any polygon on the first is to the similar and similarly described polygon on the second.



Hypothesis.—AB, CD, E are three st. lines such that AB : CD = CD : E, and L, M, similar polygons having AB, CD corresponding sides.

To prove that polygon L : polygon M = AB : E.

Proof.— $$\frac{\text{Polygon } L}{\text{Polygon } M} = \frac{AB^2}{CD^2} \qquad \text{(V—2, p. 278.)}$$

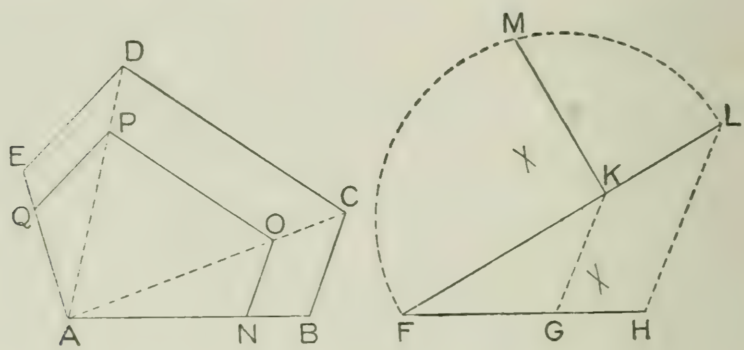
$$= \frac{AB}{CD} \cdot \frac{AB}{CD}$$

$$= \frac{AB}{CD} \cdot \frac{CD}{E} \qquad (Hyp.)$$

$$= \frac{AB}{E}.$$

Problem 3

To make a polygon similar to a given polygon and such that their areas are in a given ratio.



Let ABCDE be the given polygon and FG, GH two given st. lines.

It is required to make a polygon similar to ABCDE, and such that its area is to that of ABCDE as GH is to FG.

Construction.—Find KL a fourth proportional to FG, GH, AB. (IV—Prob. 2, p. 227.)

Find KM a mean proportional to FK, KL. (IV—Prob. 5, p. 246.)

Cut off AN = KM, and on AN construct a polygon ANOPQ similar to ABCDE.

Proof.— $\because \dfrac{\mathsf{AB}}{\mathsf{AN}} = \dfrac{\mathsf{AN}}{\mathsf{KL}}$,

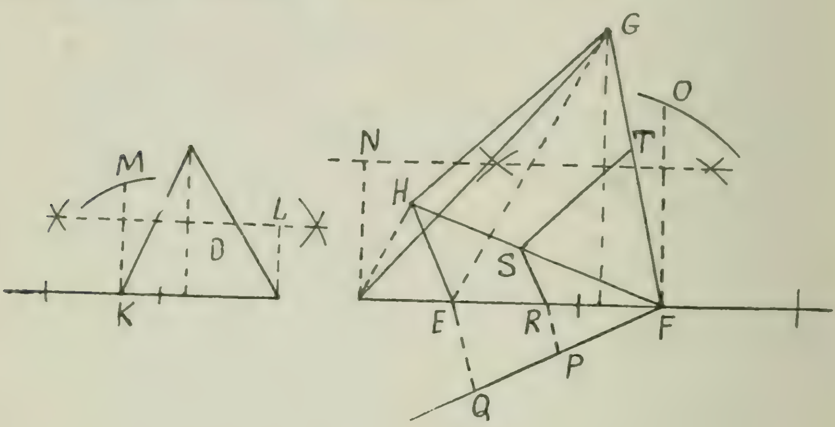
$$\therefore \frac{\text{polygon } \mathsf{ABCDE}}{\text{polygon } \mathsf{ANOPQ}} = \frac{\mathsf{AB}}{\mathsf{KL}} \qquad \text{(V—3, p. 279.)}$$

$$= \frac{\mathsf{FG}}{\mathsf{GH}}.$$

And $\therefore \dfrac{\text{polygon } \mathsf{ANOPQ}}{\text{polygon } \mathsf{ABCDE}} = \dfrac{\mathsf{GH}}{\mathsf{FG}}$.

Problem 4

To make a figure equal to one given rectilineal figure and similar to another.



Let D and EFGH be the given figures.

It is required to make a figure similar to EFGH and equal to D.

Construction.—Construct the rect. KL = D, and the rect. FN = EFGH.

Make KM the side of a square which is equal to KL, and FO the side of a square which is equal to FN; so that, $KM^2 = D$ and $FO^2 = EFGH$.

From F draw a st. line FQ and from it cut off FP = KM and FQ = FO.

Join QE, and draw PR $\parallel$ QE cutting EF at R.

On RF describe RFTS similar to EFGH.

RFTS is the required figure.

Proof.— $\because$ RFTS ||| EFGH,

$$\therefore \frac{RFTS}{EFGH} = \frac{RF^2}{EF^2}. \qquad \text{(V—1, p. 271.)}$$

$$= \frac{PF^2}{QF^2} \qquad \text{(IV—2, p. 222.)}$$

$$= \frac{KM^2}{FO^2} = \frac{D}{EFGH}.$$

$$\therefore \frac{RFTS}{EFGH} = \frac{D}{EFGH};$$

and $\therefore$ RFTS = D;

also RFTS was made similar to EFGH.

133.—Exercises

1. On a plan of which the scale is 1 inch to 2 feet, a room is represented by 30 sq. in. Find the area of the room.

2. On a map of which the scale is 4 inches to the mile, a farm is represented by 10 sq. in. Find the number of acres in the farm.

3. Construct an equilateral $\triangle$ equal in area to a given square.

4. Construct a square equal in area to a given $\triangle$.

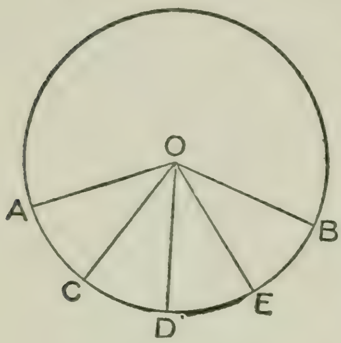
5. Construct a rectangle similar to a given rectangle and equal in area to a given square.

6. Construct a square the area of which is 15 sq. in.

7. Bisect a given $\triangle$ by a st. line drawn $\perp$ to one side.

Arcs and Angles

134.—Suppose an angle AOB at the centre of a circle to be divided into a number of equal parts AOC, COD, DOE, EOB.

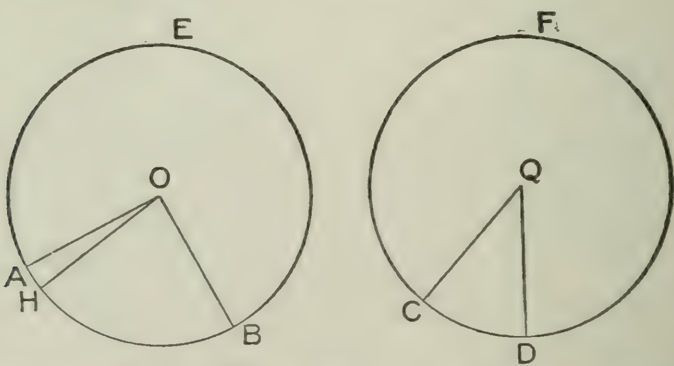


Then, by III—13, p. 167, the arcs AC, CD, DE, EB are equal to each other, and whatever multiple the angle AOB is of the angle AOC, the arc AB is the same multiple of the arc AC.

Thus, if an angle at the centre of a circle be divided into degrees and contain a of them, the arc subtending the angle will contain the arc subtending one degree a times.

Theorem 4

In equal circles, angles, whether at the centres or circumferences, are proportional to the arcs on which they stand.



Hypothesis.—In the equal circles AEB, CFD, the $\angle$s AOB, CQD at the centres stand respectively on the arcs AB, CD.

To prove that $\dfrac{\angle \mathsf{AOB}}{\angle \mathsf{CQD}} = \dfrac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{CD}}$.

Proof.—Let the $\angle$s AOB, CQD be commensurable having $\angle$ AOH a common measure. Suppose $\angle$ ACB contains $\angle$ AOH a times, and $\angle$ CQD contains $\angle$ AOH b times.

Then arc AB contains arc AH a times, and arc CD contains arc AH b times.

$$\therefore \frac{\angle \mathsf{AOB}}{\angle \mathsf{CQD}} = \frac{a \times \angle \mathsf{AOH}}{b \times \angle \mathsf{AOH}} = \frac{a}{b}.$$

And $$\frac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{CD}} = \frac{a \times \text{arc } \mathsf{AH}}{b \times \text{arc } \mathsf{AH}} = \frac{a}{b}.$$

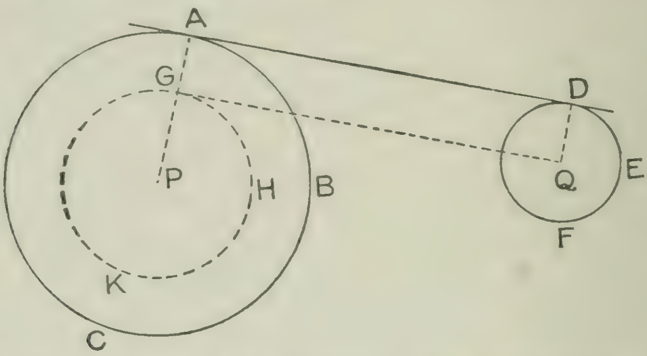
$$\therefore \frac{\angle \mathsf{AOB}}{\angle \mathsf{CQD}} = \frac{\text{arc } \mathsf{AB}}{\text{arc } \mathsf{CD}}.$$

Again, since the $\angle$s at the circumferences are respectively half the $\angle$s at the centres, on the same arcs, the $\angle$s at the circumferences are also in the ratio of the arcs on which they stand.

Analysis of a Problem—Common Tangents of Circles

135. A common method of discovering the solution of a problem begins with the drawing of the given figure or figures. The required part is then sketched in, and a careful examination is made to determine the connection between the given parts and the required result. Properties of the figure are noted, and lines are drawn that may help in finding the solution. This method of attack is known as the **Analysis of the Problem**. Its use is illustrated in the following sections.

136. **Problem.—To draw the direct common tangents to two given circles.**



Let ABC, DEF be two circles, with centres P, Q.

It is required to draw a direct common tangent to the circles ABC, DEF.

Suppose AD to be a direct common tangent touching the circles at A, D.

Join PA, QD.

PA, QD are both $\perp$ AD, and $\therefore$ PA $\parallel$ QD.

Cut off AG = DQ. Join QG.

AG is both = and $\parallel$ QD, $\therefore$ AQ is a $\parallel$gm, and as $\angle$ GAD is a rt. $\angle$, AQ is a rect.

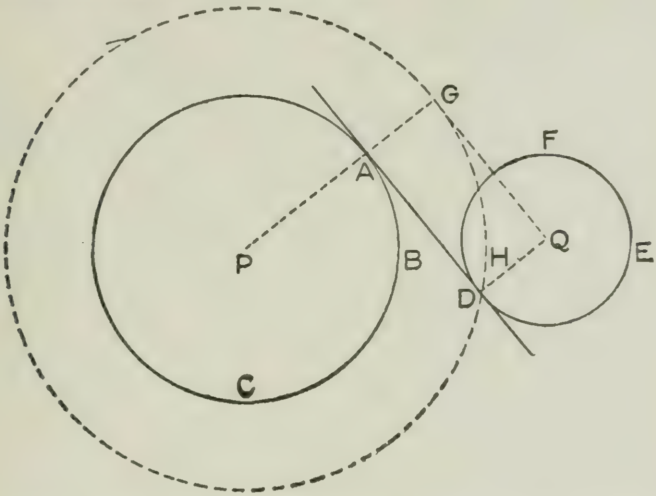
Draw a circle with centre P and radius PG.

PGQ is a rt. $\angle$, $\therefore$ QG is a tangent to the circle GHK and this tangent is drawn from the given point Q. The radius PG of the circle GHK is the difference of the radii of the given circles.

Using the construction suggested by the above analysis the pupil should make the direct drawing and prove that it is correct.

Show that two direct common tangents may be drawn.

137. **Problem.—To draw the transverse common tangents to two given circles.**



Let ABC, DEF be two circles with centres P, Q.

It is required to draw a transverse common tangent to the circles ABC, DEF.

Suppose AD to be a transverse common tangent touching the circles at A, D

Join PA, QD.

PA, QD are both $\perp$ AD, $\therefore$ PA $\parallel$ QD.

Produce PA to G making AG = DQ. Join QG.

Then AQ is seen to be a rect., and if a circle be drawn with centre P and radius PG, QG is seen to be a tangent to this circle. The radius PG of the circle GHK is the sum of the radii of the given circles.

From this analysis the pupil can make the direct construction and give the proof.

Two transverse common tangents may be drawn to the given circles.

138.—Exercises

1. Draw diagrams to show that the number of common tangents to two circles may be 4, 3, 2, 1 or 0.

2. Draw a st. line to cut two given circles so that the chords intercepted on the line may be equal respectively to two given st. lines.

3. P, Q are the centres of two circles. A common tangent (either direct or transversal) meets the line of centres at R. Show that the ratio PR : QR equals the ratio of the radii of the circles.

4. The transverse common tangents and the line of centres of two circles are concurrent.

5. The direct common tangents and the line of centres of two circles are concurrent.

6. P, Q are the centres of two circles and PA, QB any two || radii drawn in the same direction from P, Q. Show that AB produced and the direct common tangents meet the line of centres at the same point.

7. P, Q are the centres of two circles and PA, QB any two || radii drawn in opposite directions from P, Q. Show that AB and the transverse common tangents meet the line of centres at the same point.

8. Draw the direct common tangents to two equal circles.

Miscellaneous Exercises

1. Draw four circles each of radius $1\frac{3}{4}$ inches, touching a fixed circle of radius 1 inch and also touching a st. line $1\frac{1}{2}$ inches distant from the centre of the circle.

2. **DE, FG** are $\parallel$ chords of the circle **DEGF**. Prove that $DE.FG = DG^2 - DF^2$.

3. If two circles touch externally at **A** and are touched at **B, C** by a st. line, the st. line **BC** subtends a rt. $\angle$ at **A**.

4. Of all $\triangle$s of given base and vertical $\angle$, the isosceles $\triangle$ has the greatest area.

5. **ABC** is an equilateral $\triangle$ inscribed in a circle, **P** is any point on the circumference. Of the three st. lines **PA, PB, PC,** shew that one equals the sum of the other two.

6. Construct a rt.-$\angle$d $\triangle$, given the radius of the inscribed circle and an acute $\angle$ of the $\triangle$.

7. The diagonals **AC, BD** of a cyclic quadrilateral **ABCD** cut at **E.** Show that the tangent at **E** to the circle circumscribed about $\triangle$ **ABE** is $\parallel$ to **CD.**

8. **A, B, C** are three points on a circle. The bisector of $\angle$ **ABC** meets the circle again at **D. DE** is drawn $\parallel$ to **AB** and meets the circle again at **E.** Show that **DE = BC.**

9. The side of an equilateral $\triangle$ circumscribed about a circle is double the side of the equilateral $\triangle$ inscribed in the same circle.

10. **AB** is the diameter of a circle and **CD** a chord. **EF** is the projection of **AB** on **CD.** Show that **CE = DF.**

11. Construct an isosceles $\triangle$, given the base and the radius of the inscribed circle.

12. Two circles touch externally. Find the locus of the points from which tangents drawn to the circles are equal to each other.

13. Two circles, centres **C**, **D**, intersect at **A**, **B**. **PAQ** is a st. line cutting the circles at **P**, **Q**. **PC**, **QD** intersect at **R**. Find the locus of **R**.

14. Two circles touch internally at **A**; **BC**, a chord of the outer circle, touches the inner circle at **D**. Show that **AD** bisects ∠ **BAC**.

15. **P** is a given point on the circumference of a circle, of which **AB** is a given chord. Through **P** draw a chord **PQ** that is bisected by **AB**.

16. On a given base construct a △ having given the vertical ∠ and the ratio of the two sides.

17. **AB** is a given st. line and **P**, **Q** are two points such that **AP** : **PB** = **AQ** : **QB**. Show that the bisectors of ∠s **APB**, **AQB** cut **AB** at the same point.

18. **AB** is a given st. line and **P**, **Q** are two points such that **AP** : **PB** = **AQ** : **QB**. Show that the bisectors of the exterior ∠s at **P**, **Q** of the △s **APB**, **AQB** meet **AB** produced at the same point.

19. **AB** is a given st. line and **P** is a point which moves so that the ratio **AP** : **PB** is constant. The bisectors of the interior and exterior ∠s at **P** of the △ **APB**, meet **AB** and **AB** produced at **C**, **D** respectively. Show that the locus of **P** is a circle on **CD** as diameter.

20. **AB** is a st. line 2 inches in length. **P** is a point such that **AP** is twice **BP**. Construct the locus of **P**.

21. Two circles touch externally, and **A**, **B** are the points of contact of a common tangent. Show that **AB** is a mean proportional between their diameters.

22. If on equal chords segments of circles be described containing equal $\angle$s, the circles are equal.

23. Construct a quadrilateral such that the bisectors of the opposite $\angle$s meet on the diagonals.

24. Draw a circle to pass through a given point and touch two given st. lines.

25. Draw a circle to touch a given circle and two given st. lines.

26. Draw a circle to pass through two given points and touch a given circle.

27. Construct a rt.-$\angle$d $\triangle$ given the hypotenuse and the radius of the inscribed circle.

28. In $\triangle$ ABC the inscribed circle touches AB, AC at D, E respectively. The line joining A to the centre cuts the circle at F. Show that F is the centre of the inscribed circle of $\triangle$ ADE.

29. The inscribed circle of the rt.-$\angle$d $\triangle$ ABC touches the hypotenuse BC at D. Show that rect. BD.DC = $\triangle$ ABC.

30. If on the sides of any $\triangle$ equilateral $\triangle$s be described outwardly, the centres of the circumscribed circles of the three equilateral $\triangle$s are the vertices of an equilateral $\triangle$.

31. Describe three circles to touch each other externally and a given circle internally.

32. Show that two circles can be described with the middle point of the hypotenuse of a rt.-$\angle$d $\triangle$ as centre to touch the two circles described on the two sides as diameters.

33. A st. line AB of fixed length moves so as to be constantly $\parallel$ to a given st. line and A to be on the circumference of a given circle. Show that the locus of B is an equal circle.

34. Construct an isosceles $\triangle$ equal in area to a given $\triangle$ and having the vertical $\angle$ equal to one of the $\angle$s of the given $\triangle$.

35. If two chords **AB, AC,** drawn from a point **A** in the circumference of the circle **ABC,** be produced to meet the tangent at the other extremity of the diameter through **A** in **D, E** respectively, then the $\triangle$ **AED** is similar to $\triangle$ **ABC.**

36. If a st. line be divided into two parts, the sq. on the st. line equals the sum of the rectangles contained by the st. line and the two parts.

37. **ABCD** is a quadrilateral inscribed in a circle. **AB, DC** meet at **E** and **BC, AD** meet at **F.** Show that the sq. on **EF** equals the sum of the sqs. on the tangents drawn from **E, F** to the circle.

38. The st. line **AB** is divided at **C** so that **AC** = 3 **CB.** Circles are described on **AC, CB** as diameters and a common tangent meets **AB** produced at **D.** Show that **BD** equals the radius of the smaller circle.

39. **DE** is a diameter of a circle and **A** is any point on the circumference. The tangent at **A** meets the tangents at **D, E** at **B, C** respectively. **BE, CD** meet at **F.** Show that **AF** is $\parallel$ to **BD.**

40. **TA, TB** are tangents to a circle of which **C** is the centre. **AD** is $\perp$ **BC.** Show that **TB : BC = BD : DA.**

41. **ABCD** is a quadrilateral inscribed in a circle. **BA, CD** produced meet at **P,** and **AD, BC** produced meet at **Q.** Show that **PC : PB = QA : QB.**

42. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in the ratio of two given st. lines.

43. Describe a circle to pass through a given point and touch a given st. line and a given circle.

44. LMN is a rt.-∠d △ with L the rt. ∠. On the three sides equilateral △s LEM, MFN, NDL are described outwardly. LG is ⊥ MN. Prove that △ FGM = △ LEM and △ FGN = △ NDL.

45. L is the rt. ∠ of a rt.-∠d △ LMN in which LN = 2 LM. Also LX ⊥ MN. Prove that LX = $\frac{2}{5}$ MN.

46. A st. line meets two intersecting circles in P and Q, R and S and their common chord in O. Prove that OP, OQ, OR, OS, taken in a certain order, are proportionals.

47. LMN is a semi-circle of which O is the centre, and OM ⊥ LN. A chord LDE cuts OM at D. Prove that LM is a tangent to the circle MDE.

48. The bisector of ∠ F of △ FGH meets the base GH in E and the circumcircle in D. Prove that DG^2 = DE.DF.

49. POQ, ROS are two st. lines such that PO : OQ = 3 : 4 and RO : OS = 2 : 5. Compare areas of △s POR, QOS; and also areas of △s POS, QOR.

50. Trisect a given square by st. lines drawn ∥ to one of its diagonals.

51. Construct a △ having its base 8 cm., the other sides in the ratio of 3 to 2, and the vertical ∠ = 75°.

52. In two similar △s, the parts lying within the △ of the right bisectors of corresponding sides have the same ratio as the corresponding sides of the △.

53. KMN, LMN are △s on the same base and between the same ∥s. KN, LM cut at E. A line through E, ∥ MN, meets KM in F and LN in G. Prove that FE = EG.

54. Construct a △ having given the vertical ∠, the ratio of the sides containing that ∠, and the altitude drawn to the base.

55. From a point P without a circle two secants PFG, PED are drawn, and PQ drawn || FD meets GE produced at Q. Prove that PQ is a mean proportional between QE, QG.

56. LD bisects ∠ L of △ LMN and meets MN at D. From D the line DE || LM meets LN at E, and DF || LN meets LM at F. Prove that FM : EN = $LM^2 : LN^2$.

57. LMN is a △ ∠ rt.-d at L. LD ⊥ MN and meets a line drawn from M ⊥ LM at E. Prove that △ LMD is a mean proportional between △s LDN, MDE.

58. Two circles touch externally at D and PQ is a common tangent. PD and QD produced meet the circumferences at L, M respectively. Show that PM and QL are diameters of the circles.

59. The common tangent to two circles which intersect subtends supplementary ∠s at the points of intersection.

60. Two circles intersect at Q and R, and ST is a common tangent. Show that the circles described about △s STR, STQ are equal.

61. A st. line DEF is drawn from D the extremity of a diameter of a circle cutting the circumference at E and a fixed st. line ⊥ to the diameter at F. Show that the rect. DE.DF is constant for all positions of DEF.

62. A chord LM of a circle is produced to E such that ME is one-third of LM; a tangent EP is drawn to the circle and produced to D such that PD = EP. Prove that △ ELD is isosceles.

63. Draw a st. line to touch one circle and to cut another, the chord cut off being equal to a given st. line.

64. Two equal circles are placed so that the transverse common tangent is equal to the radius. Show that the tangent from the centre of one circle to the other equals the diameter of each circle.

65. Construct a $\triangle$ having its medians respectively equal to three given st. lines.

66. Construct a $\triangle$ given one side and the lengths of the medians drawn from the ends of that side.

67. Construct a $\triangle$ given one side, the median drawn to the middle point of that side, and a median drawn from one end of that side.

68. Construct a $\triangle$ having $\angle \mathsf{A} = 20°$, $\angle \mathsf{C} = 90°$, and $c - a = 4$ cm.

69. Construct a $\triangle$ having $\angle \mathsf{C} = 90°$, $b = 6$ cm., and $c - a = 3{\cdot}5$ cm.

70. Construct a $\triangle$ having $a = 7$ cm., $c - b = 3$ cm., and $\angle \mathsf{C} - \angle \mathsf{B} = 28°$.

71. If a st. line be drawn in any direction from one vertex of a $\|$gm, the $\perp$ to it from the opposite vertex equals the sum or difference of the $\perp$s to it from the two remaining vertices.

72. **PQ** is a chord of a circle $\perp$ to the diameter **LM**, and **E** is any point in **LM**. If **PE**, **QE** meet the circumference in **S**, **R** respectively, show that **PS** = **QR**; and that **RS** $\perp$ **LM**.

73. **P** is any point in a diameter **LM** of a circle, and **QR** is a chord $\|$ **LM**. Prove that $\mathsf{PQ}^2 + \mathsf{PR}^2 = \mathsf{PL}^2 + \mathsf{PM}^2$.

74. On the hypotenuse **EF** of the rt.-$\angle$d $\triangle$ **DEF** a $\triangle$ **GEF** is described outwardly having $\angle$**GEF** = $\angle$**DEF** and $\angle$**GFE** a rt. $\angle$. Prove that $\triangle$ **GFE** : $\triangle$ **DEF** = **GE** : **ED**.

75. Two quadrilaterals whose diagonals intersect at equal $\angle$s are to one another in the ratio of the rectangles contained by the diagonals.

76. P is any point in the side LM of a $\triangle$ LMN. The st. line MQ, $\parallel$ PN, meets LN produced at Q; and X, Y are points in LM, LQ respectively, such that $LX^2 = LP.LM$ and $LY^2 = LN.LQ$. Prove that $\triangle$ LXY = $\triangle$ LMN.

77. EFP, EFQ are circles and PFQ is a st. line. ER is a diameter of circle EFP and ES a diameter of EFQ. Prove $\triangle$ EPR : $\triangle$ EQS as the squares on the radii of the circles.

78. If P is the point of intersection of an external common tangent PQR to two circles with the line of centres, prove that PQ : PR as the radii of the circles. Also, if PCDEF is a secant, prove that PC : PE = PD : PF

79. A point E is taken within a quadrilateral FGHK such that $\angle$EFK = $\angle$GFH and $\angle$EKF = $\angle$GHF. GE is joined. Prove $\triangle$ FEG $|||$ $\triangle$ FHK.

80. Through a given point within a circle, draw a chord that is divided at the point in a given ratio

81. From P, a point on the circumference of a circle, tangents PE, PF are drawn to an inner concentric circle. GEFH is a chord, and PE meets the circumference at Q. Prove $\triangle$s PGF, PEH, GEQ similar; also show that $GQ^2 : GP^2 = GE : GF$.

82. L is the vertex of an isosceles $\triangle$ LMN inscribed in a circle, LRS is a st. line which cuts the base in R and meets the circle in S. Prove that $SL.RL = LM^2$.

83. PQR is a rt.-$\angle$d $\triangle$ with P the rt. $\angle$. PD $\perp$ QR; DM $\perp$ PQ and DN $\perp$ PR. Prove that $\angle$ QMR = $\angle$ QNR.

84. DEF is an isosceles $\triangle$ with $\angle$ D = 120.° Show that if EF be trisected at G and H, the $\triangle$ DGH is equilateral.

85. **AS** and **AT**, **BP** and **BQ** are tangents from two points **A** and **B** to a circle. **C**, **D**, **E**, **F** are the middle points of **AS**, **AT**, **BP**, **BQ** respectively. Prove that **CD**, **EF**, produced if necessary, meet on the right bisector of **AB**. *(Let* **O** *be the centre of the circle;* **L** *and* **M** *the points where* **OA**, **OB** *cut the chords of contact. Prove* **A**, **L**, **M**, **B** *concyclic, etc.)*

86. If from the middle point of an arc two st. lines be drawn cutting the chord of the arc and the circumference, the four points of intersection are concyclic.

87. If a st. line be divided at two given points, find a third point in the line, such that its distances from the ends of the line may be proportional to its distances from the two given points.

88. Prove geometrically that the arithmetic mean between two given st. lines is greater than the geometric mean between the two st. lines.

89. A square is inscribed in a rt.-angled triangle, one side of the square coinciding with the hypotenuse: prove that the area of the square is equal to the rectangle contained by the extreme segments of the hypotenuse.

90. Any regular polygon inscribed in a circle is the geometric mean between the inscribed and circumscribed regular polygons of half the number of sides.

91. The diagonal and the diagonals of the complements of the parallelograms about the diagonal of a parallelogram are concurrent.

92. Develop the formula for the area of a $\triangle$, $\sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$ and a, b, c are the sides.

Solution of 92. In $\triangle$ **ABC**, draw **AX** $\perp$ **BC**, and let **AX** $= h$, **BX** $= x$. Then **CX** $= a - x$.

Area of $\triangle$ **ABC** $= \frac{1}{2} a h$.

$$h^2 = b^2 - (a - x)^2 = c^2 - x^2,$$

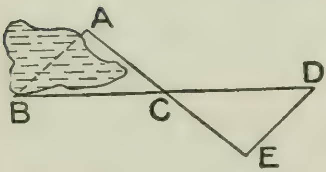
$$\therefore \quad x = \frac{a^2 - b^2 + c^2}{2a}.$$

$$h^2 = c^2 - \frac{(a^2 - b^2 + c^2)^2}{4a^2}.$$

$$\begin{aligned} 4a^2 h^2 &= 4a^2 c^2 - (a^2 - b^2 + c^2)^2 \\ &= (2ac + a^2 - b^2 + c^2)(2ac - a^2 + b^2 - c^2) \\ &= \{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\} \\ &= (a + b + c)(a - b + c)(a + b - c)(b - a + c) \\ &= 2s(2s - 2b)(2s - 2c)(2s - 2a). \end{aligned}$$

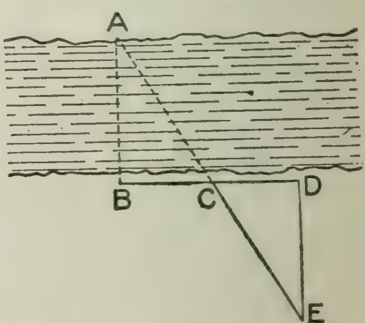
$$\therefore \tfrac{1}{4} a^2 h^2 = s(s - a)(s - b)(s - c),$$

$$\text{And } \tfrac{1}{2} ah = \sqrt{s(s - a)(s - b)(s - c)}.$$



93. Show from the diagram how the distance between two points, **A, B** at opposite sides of a pond may be found by measurements on land.

94. Show from the diagram how the breadth of a river may be found by measurements made on one side of it.



95. Given a st. line **AB**, construct a continuation of it **CD**, **AB** and **CD** being separated by an obstacle.

96. **AB, CD** are two lines which would meet off the paper. Draw a st. line which would pass through the point of intersection of **AB, CD**, and bisect the $\angle$ between them.

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(1) a pl proposed to produce y P making PD = PY

(2) another proposed to produce y P to D making y D = MB

both solutions are possible

Teach